Mon. Not. R. Astron. Soc. **000**, 000–000 (0000) Printed 1 February 2008 (MN LaTeX style file v1.4)

Non-linear evolution of the tidal angular momentum of protostructures II: non-Gaussian initial conditions

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arXiv:astro-ph/9704119v1 14 Apr 1997

1 February 2008

ABSTRACT

The formalism that describes the non-linear growth of the angular momentum **L** of protostructures from tidal torques in a Friedmann Universe, as developed in a previous paper, is extended to include non-Gaussian initial conditions. We restrict our analysis here to a particular class of non-Gaussian primordial distributions, namely multiplicative models. In such models, strongly correlated phases are produced by obtaining the gravitational potential via a nonlinear local transformation of an underlying Gaussian random field. The dynamical evolution of the system is followed by describing the trajectories of fluid particles using second-order Lagrangian perturbation theory. In the Einstein-de Sitter universe, the lowest-order perturbative correction to the variance of the linear angular momentum of collapsing structures grows as $t^{8/3}$ for generic non-Gaussian statistics, which contrasts with the $t^{10/3}$ growth rate characteristic of Gaussian statistics. This is a consequence of the fact that the lowest-order perturbative spin contribution in the non-Gaussian case arises from the third moment of the gravitational potential, which is identically zero for a Gaussian field. Evaluating these corrections at the maximum expansion time of the collapsing structure, we find that these non-Gaussian and non-linear terms can be as high as the linear estimate, without the degree of non-Gaussianity as quantified by skewness and kurtosis of the density field being unacceptably large. The results suggest that higher-order terms in the perturbative expansion may contribute significantly to galactic spin which contrasts with the straightforward Gaussian case.

Key words: galaxies: formation – large-scale structure of Universe

1 INTRODUCTION

It is often assumed that protostructures formed as primordial density fluctuations are amplified by gravity. The initial density field is generally assumed to be a Gaussian random field and the amplitude of fluctuations on all scales is then characterised by the density power spectrum. An appealing consequence of this gravitational instability model is that patches of matter are spun-up, as density inhomogeneities exert tidal torques on collapsing objects (Hoyle 1949), thereby providing an explanation for the spin of galaxies. The amount of angular momentum tidally induced on a given patch of matter depends mainly on its shape and scale R (mass M), and to a lesser extent also on its environment. Simple arguments based on linear gravitational instability theory lead to values for the angular momentum of the Milky Way which compare favourable to observational estimates (Peebles 1969). The *statistical* distribution of the spin of collapsing objects has been obtained by studying tidal torques on (high) peaks of the linear Gaussian density field (Hoffman 1988; Heavens & Peacock 1988), in the formalism of Peacock and Heavens (1985) and Bardeen et al. (1986; BBKS). More recently, we adopted the Lagrangian approach suggested by White (1984) and the BBKS peaks formalism to derive the probability distribution of the rms spin L (Catelan & Theuns 1996a): we showed that the distribution of this variable is practically indistinguishable from that of the modulus $|\mathbf{L}|$ as obtained by Heavens & Peacock (1988), indicating that the rms spin is well suited to characterise the statistics of rotational properties. In addition, we used the same formalism to derive the distribution of the specific angular momentum L/M versus M on galactic scales (Catelan & Theuns 1996a), which allows a direct comparison with the observational data reported in Fall (1983) as well as with the numerical simulations of Navarro, Frenk & White (1995).

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The above investigations have been restricted to the linear regime, during which the galaxy spin grows proportional to the cosmic time t (Doroshkevich 1970; White 1984). We have carried out a theoretical analysis of how the tidal angular momentum evolves during the mildly non-linear regime (Catelan & Theuns 1996b; Paper I). The formalism employs perturbative solutions of the Lagrangian dynamical equations describing the motion of the fluid, and assumes Gaussian initial conditions. The Lagrangian approach shows to be very well suited to treat the non-linear evolution of the galactic spin, because it is powerful in describing the growth of the mass-density fluctuations on the one hand (Zel'dovich 1970a, b; Buchert 1992; Bouchet et al. 1992; Catelan 1995); and on the other hand, the usual difficulty of inverting the mapping from initial Lagrangian coordinates $\mathbf{q}$ to Eulerian coordinates $\mathbf{x}$ is completely by-passed. The latter is because the angular momentum $\mathbf{L}$ is invariant with respect to an Eulerian or Lagrangian description. In spite of the fact that the leading spin corrections grow faster ($\propto t^{5/3}$) than the linear term, the main result of the perturbative investigation as reported in Paper I is that the predictions of linear theory are rather accurate in quantifying the evolution of the angular momentum of protostructures before collapse sets in: non-linear corrections increase the rms spin by a factor ~ 1.3 for the CDM power spectrum on galactic scales.

In this paper we extend the perturbative investigation of Paper I by considering non-Gaussian initial conditions. Specifically, we want to address the question whether allowing the primordial matter field to deviate from the Gaussian distribution modifies the subsequent evolution of the angular momentum during the mildly non-linear regime.

A first problem we have to face is the wide variety of possible non-Gaussian probability distributions. Primordial non-Gaussian fluctuations are predicted in many different cosmological scenarios. Topological defects remaining after an early phase-transition (Kibble 1976) such as cosmic strings (Zel'dovich 1980; Vilenkin 1981; Turok 1984; Scherrer, Melott & Bertschinger 1989), monopoles (Bennet & Rhie 1990) or global textures (Davis 1987; Turok 1989; Turok & Spergel 1990) are examples of models whose statistics are not Gaussian. Modifications of the inflationary scenario leading to phase correlations on cosmologically relevant scales have been discussed (Allen, Grinstein & Wise 1987; Kofman & Pogosyan 1988; Salopek & Bond 1991; Salopek 1992): these are non-Gaussian as well. Following suggestions of Kofman et al. (1989), Moscardini et al. (1991) and Weinberg & Cole (1992) considered non-Gaussian models obtained by performing non-linear transformations of an underlying Gaussian random field. In particular, this last class of non-Gaussian models is interesting in that one is able, by tuning free model parameters, to reproduce some of the particular properties of, e.g., the texture-seeded Cold Dark Matter model (Gooding, Spergel & Turok 1991; Park, Spergel & Turok 1991) or the cosmic explosions scenario (Ostriker & Cowie 1981).

Non-Gaussian perturbations constitute a more general statistical model than Gaussian ones and can be adopted to compute cosmological observables (see Wise 1988), such as spatial galaxy correlation functions (Matarrese, Lucchin & Bonometto 1986; Scherrer & Bertschinger 1991), peculiar velocity correlation functions (Scherrer 1992; Catelan & Scherrer 1995; Moessner 1995), expected size and frequency of high density regions (Catelan, Lucchin & Matarrese 1988; Matsubara 1995), hotspots and coldspots in the cosmic microwave background radiation (Coles & Barrow 1987; Kung 1993), and higher order temperature correlation functions (Gangui et al. 1994; Moessner, Perivolaropoulos & Brandenberger 1994).

We limit our analysis in this paper to *multiplicative* non-Gaussian processes, which have been analysed recently with N-body simulations in the context of a Cold Dark Matter cosmology (Messina et al. 1990; Moscardini et al. 1991; Matarrese et al. 1991; Weinberg & Cole 1992). In such models, the gravitational potential is obtained via a local non-linear transformation of an underlying Gaussian random field. These models are interesting because the procedure provides strongly correlated phases, while at the same time preserving the standard form of the initial power spectrum. Consequently, Gaussian and non-Gaussian models with *identical power spectra* can be compared to draw conclusions on the effects of departing from the more usual assumption of a normally distributed primordial random field. Inflation-generated non-Gaussian fluctuations are expected to be of the multiplicative type (Matarrese, Ortolan & Lucchin 1989; Kofman et al. 1990; Barrow & Coles 1990). An important example of a multiplicative process is the lognormal statistic, which has recently been proposed as a simple phenomenological model for describing the (present day) non-linear density field (Coles & Jones 1991; see also Bernardeau & Kofman 1995).

In Section 2 we review how Lagrangian perturbation theory can be used to obtain corrections to the linear angular momentum due to non-Gaussian initial conditions. In Section 3 we introduce the non-Gaussian statistics that we will investigate. In the next Section, we combine the obtained results to compute the spin corrections for the different non-Gaussian statistics. Finally, Section 5 summarises our findings. Technical Appendices contain details on different aspects of the computations.

2 NON-LINEAR EVOLUTION OF THE TIDAL ANGULAR MOMENTUM

Let us assume that the behaviour of matter on scales smaller than the horizon is similar as that of a Newtonian pressureless and irrotational self–gravitating fluid embedded in an expanding universe with arbitrary density parameter Ω. For the sake of simplicity, we consider the case of zero cosmological constant and leave to the interested reader the corresponding generalisation. Additionally, luminous objects like galaxies and clusters are assumed to grow through gravitational amplification of primordial positive density fluctuations δ of the collisionless fluid.

In what follows, comoving coordinates are denoted by $\mathbf{x}$ and physical distances by $\mathbf{r} = a(t)\mathbf{x}$, where $a(t)$ is the scale factor and t the standard cosmic time. The dynamical fluid equations in a non-flat Friedmann universe simplify considerably if, instead of t, one adopts the temporal coordinate τ defined by $d\tau \equiv a^{-2}\,dt$ (Shandarin 1980). In fact, in terms of τ, the peculiar velocity and the peculiar acceleration are given by $d\mathbf{x}/d\tau \equiv \dot{\mathbf{x}} \equiv a(\tau)\,\mathbf{u}(\mathbf{x},\tau)$, and $d^2\mathbf{x}/d\tau^2 \equiv \ddot{\mathbf{x}} \equiv \mathbf{g}(\mathbf{x},\tau)$. As discussed in Shandarin (1980), this dimensionless time τ is negative and the initial cosmological singularity at $t = 0$ corresponds to $\tau = -\infty$. Furthermore, the infinity of the cosmic time, $t = +\infty$, in the open models corresponds to $\tau = -1$, whereas $t = +\infty$ in the critical Einstein-de Sitter universe corresponds to $\tau = 0$. Finally, the contraction phase in the closed models starts at $\tau = 0$. In terms of the density parameter Ω, one has $\tau = -\sqrt{-k}\,(1-\Omega)^{-1/2}$, where k is the curvature constant ($k = -1$ for open universes and $k = 1$ for closed universes). The case $\Omega = 1\,(k = 0)$ is a singular point for the latter transformation and in this case we take $\tau \equiv -(3t)^{-1/3}$, which corresponds to using $a(t) \equiv (3t)^{2/3}$ or $t_0 = t/a^{3/2} \equiv 1/3$, so defining the unit of time. The scale factor $a(\tau)$ may then be written for all Friedmann models as $a(\tau) = (\tau^2 + k)^{-1}$.

The linear evolution of angular momentum of proto-objects is most easily analysed using the Zel'dovich (1970a, b) formulation (see White 1984; Catelan & Theuns 1996a) and the mildly non-linear spin growth is most easily analysed using Lagrangian perturbation theory. We recall that the Zel'dovich approximation coincides with the linear Lagrangian description.

We will essentially adopt the formulation of the Lagrangian gravitational theory for a collisionless Newtonian fluid as presented in Catelan (1995; see also references therein) but note that in the present paper, as we did in Paper I as well, the variable τ has opposite sign and the growth factor of the linear density perturbation is normalised differently. An alternative formulation of the Lagrangian theory may be found in Buchert (1992).

2.1 Basic tool: Lagrangian theory

In the Lagrangian formulation, the departure at time τ of mass elements from their initial position $\mathbf{q}$ is described in terms of the displacement vector field $\mathbf{S}$,

$$\mathbf{x}(\mathbf{q},\tau) \equiv \mathbf{q} + \mathbf{S}(\mathbf{q},\tau)\ . \tag{1}$$

The trajectory $\mathbf{S}(\mathbf{q},\tau)$ satisfies the Lagrangian 'irrotationality' condition and the Poisson equation given by (Catelan 1995)

$$\epsilon_{\alpha\beta\gamma}\,\left[(1+\nabla\cdot\mathbf{S})\,\delta_{\beta\sigma} - S_{\beta\sigma} + S^C_{\beta\sigma}\right]\dot{S}_{\gamma\sigma} = 0\ , \tag{2}$$

$$\left[(1+\nabla\cdot\mathbf{S})\,\delta_{\alpha\beta} - S_{\alpha\beta} + S^C_{\alpha\beta}\right]\ddot{S}_{\beta\alpha} = \alpha(\tau)[J(\mathbf{q},\tau)-1]\ , \tag{3}$$

respectively, where $\epsilon_{\alpha\beta\gamma}$ is the totally antisymmetric Levi-Civita tensor of rank three ($\epsilon_{123} \equiv 1$), the symbol $\delta_{\alpha\beta}$ indicates the Kronecker tensor, and summation over repeated Greek indices (where $\alpha = 1, 2, 3$) is understood. In these equations, $\alpha(\tau) \equiv 6a(\tau)$ and $J \equiv 1/(1+\delta)$ is the determinant of the Jacobian of the mapping $\mathbf{x} \to \mathbf{q}$. This determinant J is non-zero until the first occurrence of shell-crossing (see, e.g., Shandarin & Zel'dovich 1989). In addition, $S_{\alpha\beta} \equiv \partial S_\alpha/\partial q_\beta$, $\nabla \equiv \nabla_{\mathbf{q}}$, and $S^C_{\alpha\beta}$ denotes the cofactor of $S_{\alpha\beta}$. In general, $S_{\alpha\beta}$ is not a symmetric tensor: $S_{\alpha\beta} = S_{\beta\alpha}$ if, and only if, the Lagrangian motion is longitudinal in which case $\mathbf{S}$ is the gradient of a potential.

The equations (2) and (3) are the complete set of dynamical equations for the vector field $\mathbf{S}$, which describes the trajectory of massive particles of a collisionless fluid embedded in an arbitrary Friedmann universe. We briefly summarise their perturbative solutions (up to second-order in the linear displacement) in the next subsection.

2.2 Lagrangian perturbative solutions

The master equations (2) and (3) are of third-order and non-local in the displacement $\mathbf{S}$ (see the discussion in Kofman & Pogosyan 1995) and it is undoubtedly very difficult to solve them rigorously. Perturbative solutions can be found by expanding the trajectory $\mathbf{S}$ in a series of which the leading term corresponds to the Zel'dovich approximation. Specifically, $\mathbf{S} = \Sigma_n \mathbf{S}_n$, where $\mathbf{S}_n = O(\mathbf{S}_1^n)$ is the n-th order approximation. Here, we will need $\mathbf{S}_1$ and $\mathbf{S}_2$ only. For brevity we use the symbol $\Theta(\tau) \equiv \ln[(\tau-1)/(\tau+1)]^{1/2}$ for the open universe case ($k = -1$) and $\Lambda(\tau) \equiv \mathrm{arctang}(1/\tau) = -i\Theta(i\tau)$ for the closed universe case ($k = +1$). We neglect decaying modes.

2.2.1 First-order approximation:

The first-order solution to equations (2) and (3) is separable in space and time and corresponds to the Zel'dovich approximation (Zel'dovich 1970a, b):

$$\mathbf{S}_1(\mathbf{q},\tau) = D(\tau)\,\mathbf{S}^{(1)}(\mathbf{q}) \equiv D(\tau)\,\nabla\psi^{(1)}(\mathbf{q})\ ; \tag{4}$$

the function $D(\tau)$ is the growth factor of linear density perturbations and is given by:

$$D(\tau) = \begin{cases} \frac{5}{2}\left\{1 + 3\,(\tau^2 - 1)\,[1 + \tau\,\Theta(\tau)\,]\right\} & \text{if } \Omega < 1 \\ \tau^{-2} & \text{if } \Omega = 1 \\ \frac{5}{2}\left\{-1 + 3\,(\tau^2 + 1)\,[1 - \tau\,\Lambda(\tau)\,]\right\} & \text{if } \Omega > 1\,. \end{cases} \tag{5}$$

The solution for the closed models can be obtained from that for the open models by substituting in the latter τ by $i\tau$ and reversing the sign to make the growing mode positive. Note that, in contrast to Bouchet et al. (1992) and Catelan (1995), we normalised $D(\tau)$ according to the suggestion of Shandarin (1980): the coefficient 5/2 is such that $D(\tau) \to \tau^{-2}$ in the limit $\tau \to -\infty$, which coincides with the Einstein-de Sitter case. $D(\tau)$ for the different universes is plotted in Fig. 1 of Shandarin (1980) and in Fig. A1 of Paper I. The function $\psi^{(1)}(\mathbf{q})$ is the (initial) gravitational potential. For later use we define its Fourier transform, $\tilde{\psi}^{(1)}(\mathbf{p}) = \int d\mathbf{q}\,\psi^{(1)}(\mathbf{q})\,\mathrm{e}^{-i\mathbf{p}\cdot\mathbf{q}}$, where $\mathbf{p}$ is the comoving Lagrangian wave vector. The Fourier transform of the linear density field, $\tilde{\delta}^{(1)}(\mathbf{p},\tau) = D(\tau)\tilde{\delta}_1(\mathbf{p})$, is related to $\tilde{\psi}^{(1)}(\mathbf{p})$ via the Poisson equation, $\tilde{\psi}^{(1)}(\mathbf{p}) = p^{-2}\tilde{\delta}_1(\mathbf{p})$.

2.2.2 *Second-order approximation*

The second-order solution is also separable and describes a longitudinal motion in Lagrangian space:

$$\mathbf{S}_2(\mathbf{q},\tau) = E(\tau)\,\mathbf{S}^{(2)}(\mathbf{q}) \equiv E(\tau)\,\nabla\psi^{(2)}(\mathbf{q})\,. \tag{6}$$

The second order growth rate $E(\tau)$ is

$$E(\tau) = \begin{cases} -\frac{25}{8} - \frac{225}{8}\,(\tau^2 - 1)\left\{1 + \tau\,\Theta(\tau) + \frac{1}{2}\left[\tau + (\tau^2 - 1)\,\Theta(\tau)\,\right]^2\right\} & \text{if } \Omega < 1 \\ -\frac{3}{7}\tau^{-4} & \text{if } \Omega = 1 \\ -\frac{25}{8} + \frac{225}{8}\,(\tau^2 + 1)\left\{1 - \tau\,\Lambda(\tau) - \frac{1}{2}\left[\tau - (\tau^2 + 1)\,\Lambda(\tau)\,\right]^2\right\} & \text{if } \Omega > 1\,. \end{cases} \tag{7}$$

The second-order non-flat solution was derived previously by Bouchet et al. (1992). The extra factor 25/4 of the present formulation is due to the different normalisation of the first-order solution D. An excellent approximation of the second-order growing mode is $E \approx -\frac{3}{7}D^2$ (see e.g. Fig. 5 of paper I). In the limit $\tau \to -\infty$ one has $E = -\frac{3}{7}\tau^{-4}$ which corresponds to the flat case. The Fourier transform of the second-order potential $\psi^{(2)}$ is (Catelan 1995):

$$\tilde{\psi}^{(2)}(\mathbf{p}) = -\frac{1}{p^2}\int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{(2\pi)^6}\left[(2\pi)^3\delta_D(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p})\right]\kappa^{(2)}(\mathbf{p}_1,\mathbf{p}_2)\,\tilde{\psi}^{(1)}(\mathbf{p}_1)\,\tilde{\psi}^{(1)}(\mathbf{p}_2)\,, \tag{8}$$

where we have defined the symmetric kernel

$$\kappa^{(2)}(\mathbf{p}_1,\mathbf{p}_2) \equiv \frac{1}{2}\left[\,p_1^2\,p_2^2 - (\mathbf{p}_1\cdot\mathbf{p}_2)^2\right]\,, \tag{9}$$

which describes the second order non-linear corrections to the trajectory of the fluid elements.

The perturbative expansion for $\mathbf{S}$ corresponds to a Taylor series in the variable $D(\tau) = \tau^{-2}$ in the Einstein-de Sitter universe but this is no longer rigorously true in a non-flat universe. However, since the higher-order growth factors can be approximated exceedingly well by powers of D, the expansion in the non-flat case is still 'close' to a Taylor expansion (see the discussion in Paper I). We proceed by considering how corrections to the linear displacement $\mathbf{S}_1$ translate into corrections to the linear angular momentum.

2.3 Non-linear spin dynamics

In this section, we briefly summarise the perturbative approach to the spin evolution. As stressed in Paper I, the angular momentum of the matter contained in the volume $V(\tau)$ at a given time τ may be written equivalently as an integral over the corresponding *initial* volume Γ:

$$\mathbf{L}(\tau) = \eta_0\int_\Gamma d\mathbf{q}\,[\mathbf{q} + \mathbf{S}(\mathbf{q},\tau)] \times \frac{d\mathbf{S}(\mathbf{q},\tau)}{d\tau}\,. \tag{10}$$

Here, $\eta_0 \equiv a^3\rho_b$, where ρ_b is the mean background density. This procedure enables us to apply the Lagrangian description of Newtonian gravity previously reviewed. The linear regime (Zel'dovich approximation) has been fully analysed in this way by Doroshkevich (1970), White (1984) and Catelan & Theuns (1996a), whereas its Eulerian counterpart was studied extensively by Heavens and Peacock (1988). We can extend the linear Lagrangian analysis of the evolution of the angular momentum $\mathbf{L}(\tau)$ to the non-linear regime by applying perturbation theory to equation (10). Perturbative corrections to $\mathbf{S}(\mathbf{q},\tau)$ (Bouchet et al. 1992; Buchert 1994; Catelan 1995 and references therein) then give perturbative corrections to $\mathbf{L}(\tau)$: formally

$$\mathbf{L}(\tau) = \sum_{h=0}^{\infty}\mathbf{L}^{(h)}(\tau) \equiv \sum_{h=0}^{\infty}\sum_{j=0}^{h}\eta_0\int_\Gamma d\mathbf{q}\;\mathbf{S}_j(\mathbf{q},\tau) \times \frac{d\mathbf{S}_{h-j}(\mathbf{q},\tau)}{d\tau}\,, \tag{11}$$

with $\mathbf{S}_0 \equiv \mathbf{q}$, hence $\mathbf{L}^{(0)} = \mathbf{0}$. To calculate the lowest-order corrections to the ensemble average $\langle \mathbf{L}^2 \rangle$, we need to compute corrections to $\mathbf{L}$ up to second-order. After reviewing briefly the results of the linear theory, we summarise the final expression of the correction $\mathbf{L}^{(2)}$.

2.3.1 Linear approximation

The linear Lagrangian theory corresponds to the Zel'dovich approximation and the first-order term in equation (11) is given by:

$$\mathbf{L}^{(1)}(\tau) = \eta_0 \, \dot{D}(\tau) \int_\Gamma d\mathbf{q} \, \mathbf{q} \times \nabla \psi^{(1)}(\mathbf{q}) \; . \tag{12}$$

Assuming that $\psi^{(1)}(\mathbf{q})$ can be adequately represented in the volume Γ by the first three terms of the Taylor series about the origin, $\mathbf{q} = \mathbf{0}$, each component $L_\alpha^{(1)}(t)$ may be written in compact form as (White 1984; Catelan & Theuns 1996a):

$$L_\alpha^{(1)}(t) = \dot{D}(\tau) \, \epsilon_{\alpha\beta\gamma} \, \mathcal{D}_{\beta\sigma}^{(1)} \, \mathcal{J}_{\sigma\gamma} = -\dot{D}(\tau) \, \epsilon_{\alpha\beta\gamma} \, \mathcal{J}_{\sigma\gamma} \int \frac{d\mathbf{p}}{(2\pi)^3} \, p_\sigma \, p_\gamma \, \widetilde{W}(pR) \, \tilde{\psi}^{(1)}(\mathbf{p}) \; , \tag{13}$$

where we introduced the deformation tensor at the origin, $\mathcal{D}_{\beta\sigma}^{(1)} \equiv \mathcal{D}_{\beta\sigma}^{(1)}(\mathbf{0}) = \partial_\beta \partial_\sigma \psi^{(1)}(\mathbf{0})$, and the inertia tensor of the mass contained in the volume Γ, $\mathcal{J}_{\sigma\gamma} \equiv \eta_0 \int_\Gamma d\mathbf{q} \, q_\sigma \, q_\gamma$. In addition, the field $\psi^{(1)}$ is now assumed to be filtered on scale R using the smoothing function W_R, whose Fourier transform is $\widetilde{W}(pR)$. Equation (13) shows that the linear angular momentum $\mathbf{L}^{(1)}$ is in general non–zero because the principal axes of the inertia tensor $\mathcal{J}_{\alpha\beta}$, which depend only on the (irregular) shape of the volume Γ, are not aligned with the principal axes of the deformation tensor $\mathcal{D}_{\alpha\beta}^{(1)}$, which depend on the location of neighbour matter fluctuations. The temporal growth of the tidal angular momentum induced by this misalignment is completely contained in the function $\dot{D}(\tau)$, which behaves as $\dot{D}(\tau) = -2\tau^{-3} \sim t$ in the Einstein–de Sitter universe, as first noted by Doroshkevich (1970). Finally, if Γ is a spherical Lagrangian volume, then $L_\alpha^{(1)} \sim \epsilon_{\alpha\beta\gamma} \, \mathcal{D}_{\beta\gamma}^{(1)} = 0$. Consequently, the matter contained initially in a spherical volume does not gain any tidal spin during the linear regime (see also the discussion in White 1984).

2.3.2 Second-order approximation

The second-order term in equation (11) involves the second-order displacement $\mathbf{S}^{(2)}$:

$$\mathbf{L}^{(2)}(\tau) = \eta_0 \int_\Gamma d\mathbf{q} \, \mathbf{q} \times \frac{d\mathbf{S}_2}{d\tau} = \eta_0 \, \dot{E}(\tau) \int_\Gamma d\mathbf{q} \, \mathbf{q} \times \nabla \psi^{(2)}(\mathbf{q}) \; . \tag{14}$$

Note that, since $E \propto \tau^{-4}$, one has $\dot{E} \propto \tau^{-5}$ hence the second-order terms grows $\propto t^{5/3}$ in the Einstein–de Sitter universe. This growth rate was first derived by Peebles (1969). If we represent $\psi^{(2)}(\mathbf{q})$ in Γ by the first three terms of a Taylor series, as we did before for $\psi^{(1)}$, we obtain for the α-component (see Paper I):

$$\begin{aligned} L_\alpha^{(2)}(\tau) &= \dot{E}(\tau) \, \epsilon_{\alpha\beta\gamma} \, \mathcal{D}_{\beta\sigma}^{(2)} \, \mathcal{J}_{\sigma\gamma} \\ &= \dot{E}(\tau) \, \epsilon_{\alpha\beta\gamma} \, \mathcal{J}_{\sigma\gamma} \int \frac{d\mathbf{p}_1 \, d\mathbf{p}_2}{(2\pi)^6} \, \frac{(\mathbf{p}_1 + \mathbf{p}_2)_\sigma \, (\mathbf{p}_1 + \mathbf{p}_2)_\gamma}{|\mathbf{p}_1 + \mathbf{p}_2|^2} \, \widetilde{W}(|\mathbf{p}_1 + \mathbf{p}_2|R) \, \kappa^{(2)}(\mathbf{p}_1, \mathbf{p}_2) \, \tilde{\psi}^{(1)}(\mathbf{p}_1) \, \tilde{\psi}^{(1)}(\mathbf{p}_2) \; , \end{aligned} \tag{15}$$

where $\mathcal{D}_{\beta\sigma}^{(2)} \equiv \mathcal{D}_{\beta\sigma}^{(2)}(\mathbf{0}) = \partial_\beta \partial_\sigma \psi^{(2)}(\mathbf{0})$ is the second-order deformation tensor. In addition, the second-order potential $\psi^{(2)}$ in equation (15) is now explicitely smoothed on scale R by the filter $\widetilde{W}(pR)$. The filtering of the field $\psi^{(2)}$ on an appropriate scale reflects the restriction to the mildly non-linear evolution of the proto-object, since the strongly non-linear couplings between different modes are filtered out. Note that the non-linear dynamical evolution modifies only the deformation tensor and not the inertia tensor. Furthermore, if Γ is a sphere, then again $\mathbf{L}^{(2)} = \mathbf{0}$. This is in contrast to Eulerian perturbation theory since the angular momentum of an Eulerian sphere does grow in second-order perturbation theory (Peebles 1969; White 1984).

For a single collapsing region enclosed in a volume Γ it is enough to evaluate equation (11) at the time of maximum expansion τ_M to compute its final angular momentum. After τ_M the angular momentum essentially stops growing since the collapsed object is much less sensitive to external tidal couplings (Peebles 1969). However, highly non-linear interactions typically occuring after the maximum expansion time may lead to a significant redistribution of angular momentum in the final object in a complicated way (see discussion in Catelan & Theuns 1996a and references therein).

In order to compare the theory against statistical results obtained from N–body simulations or from observations, it is useful to compute the variance of the angular momentum of the object for an *ensemble* of realisations of the gravitational potential random field $\psi^{(1)}$. This programme is carried out in the next section.

3 NON-GAUSSIAN RANDOM FIELDS: SPIN ENSEMBLE AVERAGES

We simplify the previous results by considering the expectation value over the ensemble of realisations of the non-Gaussian random field $\psi^{(1)}$ of the square of $\mathbf{L}$, $\langle \mathbf{L}^2 \rangle$. Note that one can relatively easily obtain the probability distribution for the *linear* angular momentum (Heavens & Peacock 1988; Catelan & Theuns 1996a). However, such a calculation would be far more complicated for the non-linear contributions, even in the Gaussian case. We will follow the same procedure adopted in Paper I, i.e. we neglect the correlations between the inertia tensor and the gravitational field $\psi^{(1)}$.

Taking into account the mildly non-linear corrections one has

$$\langle \mathbf{L}^2 \rangle = \langle \mathbf{L}^{(1)2} \rangle + 2 \langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle + O(\tau^{-9}) \; . \tag{16}$$

The linear term $\langle \mathbf{L}^{(1)2} \rangle$ has been discussed extensively by Catelan & Theuns (1996a). Since it involves only the second-order moment of the underlying distribution (i.e. the power spectrum), its formal expression is valid for all gravitational potential fields, both Gaussian and non-Gaussian. We report its explicit expression below (see Eq. 49). Furthermore, it is easy to understand from equations (13) and (15) that $\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle$ is the lowest-order perturbative correction since it corresponds to an integral over the bispectrum $B_\psi^{(1)}$ of the initial gravitational potential field $\psi^{(1)}$:

$$(2\pi)^3 \, \delta_D(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \, B_\psi^{(1)}(\mathbf{p}_1, \mathbf{p}_2) \equiv \langle \tilde{\psi}^{(1)}(\mathbf{p}_1) \, \tilde{\psi}^{(1)}(\mathbf{p}_2) \, \tilde{\psi}^{(1)}(\mathbf{p}_3) \, \rangle \; . \tag{17}$$

Here, δ_D is the Dirac delta function. Specifically, one gets

$$\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle = \dot{D}(\tau) \, \dot{E}(\tau) \, \epsilon_{\alpha\beta\gamma} \, \epsilon_{\alpha\beta'\gamma'} \, \mathcal{J}_{\sigma\gamma} \, \mathcal{J}_{\sigma'\gamma'} \, \langle \, \mathcal{D}^{(1)}_{\beta\sigma} \mathcal{D}^{(2)}_{\beta'\sigma'} \rangle \; , \tag{18}$$

where

$$\begin{aligned} \langle \, \mathcal{D}^{(1)}_{\beta\sigma} \mathcal{D}^{(2)}_{\beta'\sigma'} \rangle \; &= \; - \int \frac{d\mathbf{p}_1 \, d\mathbf{p}_2}{(2\pi)^6} \, \frac{(\mathbf{p}_1 + \mathbf{p}_2)_\beta \, (\mathbf{p}_1 + \mathbf{p}_2)_\sigma \, (\mathbf{p}_1 + \mathbf{p}_2)_{\beta'} \, (\mathbf{p}_1 + \mathbf{p}_2)_{\sigma'}}{|\mathbf{p}_1 + \mathbf{p}_2|^2} \, \kappa^{(2)}(\mathbf{p}_1, \mathbf{p}_2) \left[\widetilde{W}(|\mathbf{p}_1 + \mathbf{p}_2| R) \right]^2 B_\psi^{(1)}(\mathbf{p}_1, \mathbf{p}_2) \\ &\equiv \; \mathcal{K}^{(2)}_{\beta\sigma\beta'\sigma'} \oplus B_\psi^{(1)} \; , \end{aligned} \tag{19}$$

which defines the kernel $\mathcal{K}^{(2)}_{\beta\sigma\beta'\sigma'}$ describing the effects of the 2-order dynamics. To maintain a compact notation, we have introduced the operation "$\oplus$" defined for integrable functions $\mathcal{F}$ and $\mathcal{G}$ as

$$\mathcal{F} \oplus \mathcal{G} \equiv \int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{(2\pi)^6} \mathcal{F}(\mathbf{p}_1, \mathbf{p}_2) \, \mathcal{G}(\mathbf{p}_1, \mathbf{p}_2) \; . \tag{20}$$

Saturating the tensors' indices in equation (18) and using the isotropy of the universe, it is possible to write equation (18) in full generality as (see Appendix A for details)

$$\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle = \frac{2}{15} \, \dot{D}(\tau) \dot{E}(\tau) \, (\mu_1^2 - 3\mu_2) \, \Sigma(R) \; , \tag{21}$$

where

$$\Sigma(R) \equiv 15 \, \langle \mathcal{D}^{(1)}_{xy} \, \mathcal{D}^{(2)}_{xy} \rangle = 15 \, \mathcal{K}^{(2)}_{xyxy} \oplus B_\psi^{(1)} \; . \tag{22}$$

Here, μ_1 and μ_2 are the first and the second invariants of the inertia tensor $\mathcal{J}$ (section 3.2 below). The factorisation in terms of the invariant $\mu_1^2 - 3\mu_2$ is typical of spin ensemble averages, both in linear and mildly non-linear regime, as far as one investigates the rotational properties of objects with preselected inertia tensor. We remark that the lowest-order perturbative correction to $\langle \mathbf{L}^{(1)2} \rangle$ grows as $\dot{D}(\tau) \, \dot{E}(\tau) \propto \tau^{-8} \propto t^{8/3}$ which contrasts with the Gaussian case where the growth rate is $t^{10/3}$ instead (cfr. Paper I).

Given a particular probability distribution for the linear gravitational potential $\psi^{(1)}$, one can compute the explicit value of $\Sigma(R)$, which involves an integral over the bispectrum. Alternatively, one can use Poisson's equation to obtain a relation between the potential and density bispectrum and compute the integral over the density bispectrum $B_\delta^{(1)}$ instead. Indeed, Poisson's equation implies the following relation between the two spectra

$$B_\delta^{(1)}(\mathbf{p}_1, \mathbf{p}_2) = p_1^2 \, p_2^2 \, |\mathbf{p}_1 + \mathbf{p}_2|^2 \, B_\psi^{(1)}(\mathbf{p}_1, \mathbf{p}_2) \; . \tag{23}$$

The latter expression is useful whenever a non-Gaussian distribution is assumed directly for the density field, rather than for the underlying gravitational potential field. For example, an interesting such case is the lognormal density distribution which was recently proposed by Coles and Jones (1991) as a reliable empirical fit for the present day density distribution function.

Before performing the detailed calculations of the non-linear corrections to the spin, we need to describe the properties of those non-Gaussian fields which we intend to investigate. Unfortunately, this next section turns out to be rather long and readers not interested in the murky details can skip directly to subsection 3.2, where the final expressions for the spin averages are summarised.

3.1 Non-Gaussian initial conditions

We will consider non-Gaussian statistics that are obtained from non-linear local transformations of an underlying Gaussian random field. These statistics are assumed either for the potential or directly for the density field and we will start with the first case. We restrict ourselves to the same non–Gaussian statistics numerically explored by Moscardini et al. (1991; see also Matarrese et al. 1991; Messina et al. 1992; Moscardini et al. 1993), namely the *lognormal* (hereafter LN) and the *Chi–squared* (hereafter χ^2) model. These statistics are imposed on the primordial potential, i.e. before the linear transfer function has transformed the initial scale free power spectrum to its present day shape. As an example, we will specialise to the Cold Dark Matter (CDM) transfer function, although our method is not restricted to CDM.

This primordial potential is a gauge–invariant (up to a zero–mode) variable whose statistics are in principle fixed by primordial processes, e.g. in the inflationary scenario. This potential is not required to have zero mean, since only its gradient is physically meaningful. This allows for a wider variety of underlying statistics than if the probability distribution were to be imposed directly on the fluctuating density field, since the latter is required to have zero mean and in addition needs to be positive, $\rho > 0$, in each point.

The models we consider are obtained by the following procedure: $\psi^{(1)}$ is found by convolving a stationary random field $\varphi(\mathbf{q})$ (the primordial potential) with a real function $\mathcal{T}(\mathbf{q})$ (the transfer function):

$$\psi^{(1)}(\mathbf{q}) \equiv \int d\mathbf{q}'\, \mathcal{T}(\mathbf{q}' - \mathbf{q})\, \varphi(\mathbf{q}') \,. \tag{24}$$

The primordial potential φ is calculated by applying some non–linear operation on an underlying Gaussian random process G, e.g. $\varphi \propto G^2$ for χ^2 and $\varphi \propto \exp(G)$ for LN statistics. The transfer function $\mathcal{T}(\mathbf{q})$ is characterised by its Fourier transform:

$$\mathcal{T}(\mathbf{p}) \equiv \int d\mathbf{q}\, \mathcal{T}(\mathbf{q})\, \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{q}} \equiv T(p)\, F_\varphi(p)^{-1/2} \,, \tag{25}$$

where $T(p)$ is the CDM transfer function (e.g. Efstathiou 1990)

$$T(p) = \left[1 + \left(ap + (bp)^{3/2} + (cp)^2\right)^\nu\right]^{-1/\nu} \,, \tag{26}$$

with $a = 6.4\,(\Omega h^2)^{-1}$ Mpc, $b = 3.0\,(\Omega h^2)^{-1}$ Mpc, $c = 1.7\,(\Omega h^2)^{-1}$ Mpc, and $\nu = 1.13$. The function $F_\varphi(p)$ is a positive correction factor applied to obtain the required CDM power spectrum for $\psi^{(1)}$ when the power spectrum of φ is assumed to be scale free:

$$P_\psi(p) = \left[\mathcal{T}(p)\right]^2 P_\varphi(p) \equiv A\, p^{-3} \left[T(p)\right]^2 \,, \tag{27}$$

and $P_\psi(p) \sim p^{-3}$ for $p \to 0$. The latter relation also defines the normalisation constant A, which has to be appropriately calculated in each model since it depends on the underlying spectrum $P_\varphi(p)$. We will show that it is possible to factorise the dependence as $P_\varphi(p) \approx A_\varphi F_\varphi(p)\, p^{-3}$, from which we obtain $A \approx A_\varphi\, F_\varphi(p)^{-1} F_\varphi(p) = A_\varphi$, where the scale-dependence introduced by the function F_φ cancels out. Consequently, the normalisation of ψ follows directly from the amplitude of φ.

The assumption implicit in equation (24) is that the statistics of $\psi^{(1)}$ are of primordial origin, i.e., they refer to the time when the fluctuations were outside the Hubble radius. In addition, the overall sign has to be fixed by the primordial physical mechanism inducing the fluctuation field. The latter statistic is left invariant in the linear regime, independent of the subsequent processes which modify the wave content of $\psi^{(1)}$ on all scales smaller than the Hubble radius.

Note that both χ^2 and LN statistics belong to the same general class $\varphi(\mathbf{q}) \propto |1 + \alpha^{-1} G(\mathbf{q})|^\alpha$; in fact, for $\alpha = 2$ and $\langle G^2 \rangle^{1/2} \equiv \sigma_G \gg 1$, the χ^2 distribution is recovered, while $\alpha \to \infty$ yields the lognormal one. For small wave numbers, where $\mathcal{T}(p) \approx 1$, the gravitational potential of scale–invariant distributions obeys the scaling law:

$$\langle \tilde{\psi}^{(1)}(\mu\mathbf{p}_1) \cdots \tilde{\psi}^{(1)}(\mu\mathbf{p}_N) \rangle\, d(\mu\mathbf{p}_1) \cdots d(\mu\mathbf{p}_N) \approx \langle \tilde{\psi}^{(1)}(\mathbf{p}_1) \cdots \tilde{\psi}^{(1)}(\mathbf{p}_N) \rangle\, d\mathbf{p}_1 \cdots d\mathbf{p}_N, \tag{28}$$

for any N, up to logarithmic corrections which arise as a consequence of the necessity to perform these integrations over a finite $\mathbf{p}$ volume. This gives a straightforward generalisation of the Harrison–Peebles–Zel'dovich scale–invariance to non–Gaussian statistics.

To compute the non-linear evolution of the tidal angular momentum we need to compute the primordial bispectrum $B_\psi^{(1)}$, which in turn is determined by the bispectrum B_φ, defined as usual. Since $\tilde{\psi}^{(1)}(p) = \mathcal{T}(p)\, \tilde{\varphi}(p)$, we find a relation between the "transferred" and the primordial potential bispectra:

$$B_\psi^{(1)}(\mathbf{p}_1, \mathbf{p}_2) = \mathcal{T}(p_1)\, \mathcal{T}(p_2)\, \mathcal{T}(|\mathbf{p}_1 + \mathbf{p}_2|)\, B_\varphi(\mathbf{p}_1, \mathbf{p}_2) \,. \tag{29}$$

It should be stressed again that the density field and the gravitational potential do not, in general, obey the same statistics. In addition, one cannot self-consistently assume any non-Gaussian distribution for the density field δ_1: for example, a Chi-squared density distribution has a positive definite 2-point correlation function, at variance with the assumed large-scale homogeneity. In contrast, the same statistics for φ do not lead to an inconsistency. Let us now describe the various models in more detail.

3.1.1 Chi–squared model

The χ^2 distribution for φ is obtained from the following transformation of the underlying Gaussian field G:

$$\varphi(\mathbf{q}) = \varphi_\circ \, G(\mathbf{q})^2 \, , \tag{30}$$

where G has zero-mean and variance σ_G^2. The power spectrum of the field G is chosen as

$$P_G(p) = p^{-3} \, \Pi(p) \, . \tag{31}$$

Here we introduced the operator $\Pi(\mathbf{p}) = \Pi(|\mathbf{p}|) = \vartheta(p - p_m)\, \vartheta(p_M - p)$, where $\vartheta(x)$ denotes the Heaviside step function. The result of projecting with $\Pi(\mathbf{p})$ in equation (31) is that $P_G(p) = p^{-3}$ in the interval $[p_m, \, p_M]$ and zero outside. The cutoff constants p_m and p_M are introduced to avoid divergences at both infra-red and ultra-violet 'wavelengths' . In terms of these cutoffs, the field G has variance $\sigma_G^2 = (2\pi^2)^{-1} \ln(p_M/p_m)$. On the other hand, the constant $\varphi_\circ$ has the same dimensions as φ and can be either negative (χ_n^2 model) or positive (χ_p^2 model). These names are chosen according to the sign of the corresponding skewness $\langle\delta^3\rangle/\langle\delta^2\rangle^{3/2}$: the χ_n^2 (χ_p^2) model is negatively (positively) skewed. Finally, it is worth noting that the flicker-noise choice for the P_G spectrum, giving equipartition in Fourier space, is stable up to negligible corrections against the non-linear transformation originating φ, both in Chi-square and LN non-Gaussian models (see below).

The Chi-squared model is an example of a scale-invariant non-Gaussian statistic (Otto et al. 1986; Lucchin & Matarrese 1988). Such models have been considered by Coles & Barrow (1987) in connection with the two-dimensional distribution of CBR fluctuations. In the inflationary context, Bardeen (1980) proposed a model where adiabatic perturbations are described by a squared Gaussian process, our χ_p^2, whose G field has a non-scale-free power spectrum.

It can be shown that the φ power spectrum corresponding to the choice (31) is (see Appendix B for details)

$$P_\varphi(p) \approx \frac{\varphi_\circ^2}{2\pi^2} \left[\beta(p) + 2\ln(1 + p/p_m) - 1\right] p^{-3} \, \Pi(p) \, , \tag{32}$$

where $\beta(p) = (1 - p/p_m)^2$, for $p_m \le p \le 2p_m$, and $\beta(p) = 1 + 2\ln(-1 + p/p_m)$, for $2p_m \le p \ll p_M$. The normalisation constant A is related to $\varphi_\circ$ as $A \approx A_\varphi = \varphi_\circ^2/2\pi^2$.

Relevant to our investigation is the expression for the linear gravitational bispectrum B_φ, which turns out to be given by a convolution over the underlying spectrum P_G, namely (see Appendix C for details)

$$B_\varphi(\mathbf{p}_1, \mathbf{p}_2) = \frac{\varphi_\circ^3}{\pi^3} \int d\mathbf{p} \, P_G(p) \, P_G(|\mathbf{p}_1 + \mathbf{p}|) \, P_G(|\mathbf{p}_2 - \mathbf{p}|) \, . \tag{33}$$

Note that the statistics of the Chi-squared potential φ are independent of the variance σ_G of the underlying Gaussian field (see Appendix C). This is not the case for the LN distribution (see below), for which the departure from Gaussianity can be tuned by varying σ_G.

Finally, we define for later use the operator

$$\Pi(\mathbf{p}_1, \mathbf{p}_2) = \Pi(\mathbf{p}_1) \, \Pi(\mathbf{p}_2) \, \Pi(\mathbf{p}_1 + \mathbf{p}_2) \, , \tag{34}$$

i.e., the projection $\Pi(\mathbf{p}_1, \mathbf{p}_2)$ is zero if any of the two vectors $\mathbf{p}_1$ or $\mathbf{p}_2$ or their sum has modulus $\le p_m$ or $\ge p_M$ and is one otherwise.

3.1.2 Lognormal model

The LN distribution for φ is obtained from the following transformation of the underlying Gaussian field G:

$$\varphi(\mathbf{q}) = \varphi_\circ \, \mathrm{e}^{G(\mathbf{q})} \, , \tag{35}$$

where, as before, G has zero-mean and variance σ_G^2. Note that $\langle\varphi\rangle = \varphi_\circ \, \exp(\sigma_G^2/2)$ and there is again the freedom of assuming $\varphi_\circ$ to be negative (LN$_n$ model) or positive (LN$_p$ model).

The lognormal statistics has been analysed in the cosmological context by many authors (see e.g., Coles 1989; Coles & Jones 1990; Messina et al. 1990; Coles , Melott & Shandarin 1993; Catelan et al. 1994; Sheth 1995).

In contrast to the χ^2 case, one gets lognormal statistics which differ by varying amounts from the Gaussian distribution by tuning the variance of G: for small σ_G, the non–Gaussian character of φ is manifest only in the properties of rare high peaks, whereas for larger σ_G, the power spectrum P_φ deviates strongly from flicker–noise.

Next, we analyse the hierarchical structure of the correlation functions of the field φ. The 2-point correlation function $\langle\varphi(\mathbf{q}_1)\, \varphi(\mathbf{q}_2)\rangle - \langle\varphi\rangle^2 \equiv \xi_\varphi(q)$ is given by

$$\xi_\varphi(q) = \langle\varphi\rangle^2 \left[\mathrm{e}^{\xi_G(q)} - 1\right] \, , \tag{36}$$

where $\xi_G(q) \equiv \langle G(\mathbf{q}_1)\, G(\mathbf{q}_2)\rangle$ and $q \equiv |\mathbf{q}_1 - \mathbf{q}_2|$. The 3-point correlation function $\langle(\varphi_1 - \langle\varphi\rangle)\,(\varphi_2 - \langle\varphi\rangle)\,(\varphi_3 - \langle\varphi\rangle)\rangle \equiv \zeta_\varphi(123)$ is (see Appendix C for the derivation)

$$\zeta_\varphi(123) = \langle\varphi\rangle^{-1}\left[\xi_\varphi(12)\,\xi_\varphi(13)+\xi_\varphi(12)\,\xi_\varphi(23)+\xi_\varphi(13)\,\xi_\varphi(23)\right] + \langle\varphi\rangle^{-3}\left[\xi_\varphi(12)\,\xi_\varphi(13)\,\xi_\varphi(23)\right]\,. \tag{37}$$

The power spectrum $P_\varphi(p)$ is given by

$$P_\varphi(p) = \langle\varphi\rangle^2 \int d\mathbf{q}\left[\mathrm{e}^{\xi_G(q)}-1\right]\,\mathrm{e}^{i\mathbf{p}\cdot\mathbf{q}}\,, \tag{38}$$

and the formal expression for the bispectrum is

$$\langle\varphi\rangle^3 B_\varphi(\mathbf{p}_1,\mathbf{p}_2) = \langle\varphi\rangle^2\left\{P_\varphi(p_1)\,P_\varphi(p_2)+P_\varphi(|\mathbf{p}_1+\mathbf{p}_2|)\left[P_\varphi(p_1)+P_\varphi(p_2)\right]\right\} + \int\frac{d\mathbf{p}}{(2\pi)^3}\,P_\varphi(p)\,P_\varphi(|\mathbf{p}_1+\mathbf{p}|)\,P_\varphi(|\mathbf{p}_2-\mathbf{p}|)\,. \tag{39}$$

A remark is now appropriate. Since we are interested in comparing the results of these non-Gaussian investigation against the results from Gaussian initial conditions reported in Paper I, it is important to preserve the form of the power spectrum. On the other hand, it can be easily understood from equation (38) that the spectrum of the field φ is a complicated function of the wavevector p, even for the simple underlying spectrum $P_G(p)$ adopted in equation (31). Fortunately, the transformed field φ may be easily constrained to have the required power spectrum if we restrict ourselves to those LN distributions for which $\sigma_G^2 \le 1$. This restriction leads to some loss of generality, but the formalism becomes much simpler, and, above all, the power spectrum of the underlying Gaussian field is preserved. In this case, since $\xi_G(q)$ is a decreasing function of q, the condition $\sigma_G^2 \le 1$ implies that $|\xi_G(q)| \le 1$, and a Taylor expansion of the exponential in the integrand (38) leads to $P_\varphi(p) \approx \langle\varphi\rangle^2\,P_G(p)$. In the following we restrict ourselves to LN distributions with $\sigma_G^2 \lesssim 1$. Consequently, the power spectrum P_φ may be written as

$$P_\varphi(p) \approx \frac{2\pi^2\sigma_G^2\,\varphi_\circ^2\,\mathrm{e}^{\sigma_G^2}}{\ln(p_M/p_m)}\,p^{-3}\Pi(p)\,, \tag{40}$$

and the bispectrum as

$$\begin{aligned} B_\varphi(\mathbf{p}_1,\mathbf{p}_2) \;\approx\; & \langle\varphi\rangle^3\left[\frac{2\pi^2\sigma_G^2}{\ln(p_M/p_m)}\right]^2\Bigg[p_1^{-3}p_2^{-3}+|\mathbf{p}_1+\mathbf{p}_2|^{-3}(p_1^{-3}+p_2^{-3}) \\ & + \frac{2\pi^2\sigma_G^2}{\ln(p_M/p_m)}\int\frac{d\mathbf{p}}{(2\pi)^3}\,p^{-3}\,|\mathbf{p}_1+\mathbf{p}|^{-3}\,|\mathbf{p}_2-\mathbf{p}|^{-3}\Pi(\mathbf{p}_1,\mathbf{p})\Pi(\mathbf{p}_2,-\mathbf{p})\Bigg]\,\Pi(\mathbf{p}_1,\mathbf{p}_2)\,. \end{aligned} \tag{41}$$

Note that the bispectrum $B_\varphi(\mathbf{p}_1,\mathbf{p}_2)$ is zero outside the region $p_m \le p_{1,2} \le p_M$ and $p_m \le |\mathbf{p}_1+\mathbf{p}_2| \le p_M$. The potential ψ now has the CDM power spectrum with amplitude $A \approx 2\pi^2\sigma_G^2\,\varphi_\circ^2\,\mathrm{e}^{\sigma_G^2}\,/\ln(p_M/p_m)$.

3.1.3 "Linearised" lognormal distribution for the density field

A completely different approach may be adopted if the non-Gaussian distribution is imposed directly on the linear *density* fluctuation field δ_1, rather than on the underlying potential. Although one is not allowed to choose any non-Gaussian density distribution, as argued earlier, there are some advantages in considering models where non-Gaussianity is coded directly in the statistics of the matter distribution. An interesting case is the LN density distribution, proposed by Coles & Jones (1991) as a reliable empirical fit to the present day density distribution function. A theoretical explanation for this fact followed from the analysis of Bernardeau (1994), who computed the evolution of the one-point probability distribution function of the cosmic smoothed density induced by gravitational evolution. Indeed, his results indicate that, during the mildly non-linear regime, the density distribution is fairly well approximated by a LN statistic when the spectral index n is close to -1. For larger values of n or for smaller values of the rms σ_δ, the resulting LN distribution is close to Gaussian. The formation of large-scale structure by gravitational instability from primordial lognormal density fields has been analysed by Weinberg and Cole (1992) using N-body simulations. A further reason to consider a LN density field is motivated by the recent extension of the Hoffman-Ribak algorithm (1991) for the construction of constrained Gaussian random fields by Sheth (1995) to include also lognormal statistics.

Let us analyse more closely the complex hierarchical structure of the LN density distribution. In particular we wish to discuss the approximate expression for the 3-point LN correlation function we are allowed to adopt in our investigation. Fortunately, as we will see below, the lognormal distribution permits the constraining of the deviations of $p(\delta)$ from the Gaussian distribution uniquely in terms of the linear-regime condition. Let us suppose that the (linear) density field $\rho \equiv \rho_b[1+\delta]$ is obtained by the transformation

$$\rho \equiv \rho_b\,\mathrm{e}^{G-\sigma_G^2/2}\,, \tag{42}$$

where, as before G is an adimensional zero-mean Gaussian random field with variance σ_G^2. The mean value is given by $\langle\rho\rangle = \rho_b$. We stress the fact that equation (42) is mathematically identical to the transformation (35), but, since it involves the density field, it is physically very different.

The 2-point LN correlation function $\xi_\delta(q)$ is given by

$$\xi_\delta(q) \equiv \langle \delta_1(\mathbf{q}_1)\, \delta_1(\mathbf{q}_2)\rangle = \mathrm{e}^{\xi_G(q)} - 1 \;, \tag{43}$$

where $\xi_G(0) = \sigma_G^2$. Evaluating $\xi_\delta(q)$ at zero lag gives the variance of the LN random field δ_1:

$$\sigma_\delta^2 = \mathrm{e}^{\sigma_G^2} - 1 \;. \tag{44}$$

We stress that the equations (43) and (44) do not contain any time dependence and combine two concepts, that of "non-linear regime" and that of "deviation from Gaussianity". Since for a LN distribution the N-point ($N > 2$) functions can be expressed as a Kirkwood expansion of the 2-point functions (see Coles & Jones 1991; also below), the linear-regime condition can be cast simply in the form $\sigma_\delta^2 \ll 1$ which then also implies $\sigma_G^2 \ll 1$, i.e. the primordial LN density distribution with small fluctuations cannot deviate strongly from Gaussianity. Alternatively, if the primordial LN density distribution were to deviate strongly from the Gaussian one then the initial conditions would contain an *intrinsic* non-linear signal, unacceptable on observational grounds. Finally, since $\xi_G(q)$ is a decreasing function of q, $\sigma_G^2 \ll 1$ implies $|\xi_G(q)| \ll 1$ and consequently also $|\xi_\delta(q)| \ll 1$, which is important for the subsequent discussion.

We first show that the density power spectrum and the underlying Gaussian spectrum $P_G(p)$ are similar in the approximation $\sigma_G^2 \ll 1$. Indeed:

$$P_\delta(p) \approx P_G(p) \equiv A\, p\, \left[T(p)\right]^2 \Pi(p) \;, \tag{45}$$

where $T(p)$ is the CDM transfer function. Additionally, we derive some properties of the structure of the LN linear potential bispectrum B_δ, which is the Fourier transform of the 3-point correlation function $\zeta_\delta(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \equiv \zeta_\delta(123)$,

$$\zeta_\delta(123) = \xi_\delta(12)\, \xi_\delta(13) + \xi_\delta(12)\, \xi_\delta(23) + \xi_\delta\, \xi_\delta(23) + \xi_\delta(12)\, \xi_\delta(13)\, \xi_\delta(23) \;. \tag{46}$$

The important point to note is that the expressions (43) and (46) may be simplified if we restrict ourselves to the case $\sigma_G^2 \ll 1$, which corresponds to the linear regime, as explained earlier. Since in this case also $\xi_\delta \ll 1$, it seems reasonable to approximate the 3-point function ζ_δ by the relation

$$\zeta_\delta(123) \approx \xi_\delta(12)\, \xi_\delta(13) + \xi_\delta(12)\, \xi_\delta(23) + \xi_\delta(13)\, \xi_\delta(23) \;. \tag{47}$$

The corresponding bispectrum reduces to

$$\begin{aligned} B_\delta(\mathbf{p}_1, \mathbf{p}_2) \;&\approx\; P_\delta(p_1)\, P_\delta(p_2) + P_\delta(|\mathbf{p}_1 + \mathbf{p}_2|)\, \left[P_\delta(p_1) + P_\delta(p_2)\right] \\ &\approx\; P_G(p_1)\, P_G(p_2) + P_G(|\mathbf{p}_1 + \mathbf{p}_2|)\, \left[P_G(p_1) + P_G(p_2)\right] \;, \end{aligned} \tag{48}$$

where the cubic terms have been neglected with respect to the quadratic ones. This approximation allows a considerable simplification of the calculations, yet the loss of accuracy of the whole description is small. We stress that the same strategy cannot be applied to the LN gravitational potential since the constraint $\sigma_\varphi \ll 1$ does not imply a restriction to the linear regime.

3.2 Ensemble averages

3.2.1 Linear approximation: $\langle \mathbf{L}^{(1)2} \rangle$

The ensemble average of the rms linear angular momentum, $\langle \mathbf{L}^{(1)2} \rangle$, is computed and discussed extensively in Catelan & Theuns (1996a). Here, we briefly summarise the main results, which hold for any underlying probability distribution, i.e. for both Gaussian and non-Gaussian statistics. We have:

$$\langle \mathbf{L}^{(1)2} \rangle = \frac{2}{15} \dot{D}(\tau)^2 (\mu_1^2 - 3\mu_2)\, \sigma(R)^2 \;, \tag{49}$$

where the quantity $\sigma(R)^2$ is the mass variance on the scale R which is obtained from the power spectrum through $\sigma(R)^2 \equiv (2\pi^2)^{-1} \int_0^\infty dp\, p^6\, P_\psi(p) \widetilde{W}(pR)^2$. We will adopt a Gaussian smoothing function, $\widetilde{W}(pR) = \exp(-p^2 R^2/2)$. Equation (49) holds for any power spectrum $\langle \tilde{\psi}^{(1)}(\mathbf{p}_1)\, \tilde{\psi}^{(1)}(\mathbf{p}_2) \rangle_\psi \equiv (2\pi)^3\, \delta_D(\mathbf{p}_1 + \mathbf{p}_2)\, P_\psi(p_1)$ and the value of $\langle \mathbf{L}^{(1)2} \rangle$ depends on the normalisation of the spectrum. The general expression (49) is *independent* of the details of the boundary surface of the volume Γ but depends on μ_1 and μ_2, which are the first and the second invariant of the inertia tensor $\mathcal{J}_{\alpha\beta}$. Specifically, denoting the eigenvalues of the inertia tensor by ι_1, ι_2 and ι_3, one has $\mu_1 \equiv \iota_1 + \iota_2 + \iota_3$ and $\mu_2 \equiv \iota_1\iota_2 + \iota_1\iota_3 + \iota_2\iota_3$. For a spherical volume, $\iota_1 = \iota_2 = \iota_3$, hence $\langle \mathbf{L}^{(1)2} \rangle \propto \mu_1^2 - 3\mu_2 = 0$, as we stressed before.

3.2.2 Higher-order approximation: $\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle$

The calculation of the term $\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle$ takes advantage of the results of the second-order approximation, and essentially reduces to specialising the function $\Sigma(R)$ to the particular statistics chosen, since $\Sigma(R) = 15\, \mathcal{K}^{(2)}_{xyxy} \oplus B^{(1)}_\psi$, as explained in the introduction of Section 3. The final expressions, valid for a CDM power spectrum $P_\delta(p) \equiv Ap[T(p)]^2 = p^4\, P_\psi(p) = p^4\, [\mathcal{T}(p)]^2\, P_\varphi(p)$,

where $\mathcal{T}(p) \equiv F_\varphi(p)^{-1/2}\, T(p)$ and $P_\varphi(p) \approx A_\varphi\, F_\varphi(p)\, p^{-3}$, may be written for the different statistics as (compare with equation 21):

• For a Chi-squared gravitational field:

$$\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle = \frac{2}{15}\, \dot{D}(\tau)\dot{E}(\tau)\, (\mu_1^2 - 3\mu_2)\, \Sigma_{\chi^2}(R)\ , \tag{50}$$

where the function Σ_{χ^2} depends on the smoothing scale R and the normalisation of the spectrum A, as

$$\Sigma_{\chi^2}(R) = 15\, \mathcal{K}^{(2)}_{xyxy}(\mathbf{p}_1, \mathbf{p}_2) \oplus \mathcal{T}(p_1)\, \mathcal{T}(p_2)\, \mathcal{T}(|\mathbf{p}_1 + \mathbf{p}_2|)\, B_\varphi(\mathbf{p}_1, \mathbf{p}_2)\ . \tag{51}$$

The correction $F_\varphi(p)$ is such that [see equation (32) for the symbols]

$$\mathcal{T}(p) = T(p) \left[\beta(p) + 2\ln(1 + p/p_m) - 1\right]^{-1/2}\ , \tag{52}$$

and the normalisation constant is $A \equiv \varphi_\circ^2/2\pi^2$. The primordial bispectrum may be deduced from equation (33),

$$B_\varphi(\mathbf{p}_1, \mathbf{p}_2) = \pm (2A)^{3/2} \int d\mathbf{p}\; p^{-3}\, |\mathbf{p}_1 + \mathbf{p}|^{-3}\, |\mathbf{p}_2 - \mathbf{p}|^{-3} \Pi(\mathbf{p}_1, \mathbf{p})\, \Pi(\mathbf{p}_2, -\mathbf{p})\, \Pi(\mathbf{p}_1, \mathbf{p}_2)\ . \tag{53}$$

Typically, these integrals have to be evaluated numerically. The upper (lower) sign is for positively (negatively) skewed distributions. Note that the average $\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle$ factorises the same invariant $\mu_1^2 - 3\mu_2$ of the inertia tensor $\mathcal{J}$ as appeared in the linear term [see equation (49)]: this is a direct consequence of the isotropy of the universe which is independent of the underlying statistical properties of the matter distribution.

• For a LN gravitational potential field, the calculation may be partially done analytically to give $\Sigma_{LN}(R)$, which is a sum of three terms: $\Sigma_{LN}(R) = \Sigma^{(a)}_{LN}(R) + \Sigma^{(b)}_{LN}(R) + \Sigma^{(c)}_{LN}(R)$ which originate from the Kirkwood expansion of the bispectrum $B^{(1)}_\psi$ in equation (39):

$$\Sigma_{LN}(R) = 15\, \mathcal{K}^{(2)}_{xyxy}(\mathbf{p}_1, \mathbf{p}_2) \oplus T(p_1)\, T(p_2)\, T(|\mathbf{p}_1 + \mathbf{p}_2|)\, B_\varphi(\mathbf{p}_1, \mathbf{p}_2)\ ; \tag{54}$$

where

$$\begin{aligned} B_\varphi(\mathbf{p}_1, \mathbf{p}_2) \;\; &\approx \;\; \pm A^{3/2} \sqrt{\frac{2\pi^2\, \sigma_G^2}{\ln(p_M/p_m)}} \left[p_1^{-3} p_2^{-3} + |\mathbf{p}_1 + \mathbf{p}_2|^{-3} (p_1^{-3} + p_2^{-3}) \right. \\ &+ \;\; \left. \frac{2\pi^2\, \sigma_G^2}{\ln(p_M/p_m)} \int \frac{d\mathbf{p}}{(2\pi)^3}\, p^{-3}\, |\mathbf{p}_1 + \mathbf{p}|^{-3}\, |\mathbf{p}_2 - \mathbf{p}|^{-3} \Pi(\mathbf{p}_1, \mathbf{p}) \Pi(\mathbf{p}_2, -\mathbf{p}) \right] \Pi(\mathbf{p}_1, \mathbf{p}_2)\, . \end{aligned} \tag{55}$$

In this expression, the product $p_1^{-3} p_2^{-3}$ gives rise to the $\Sigma^{(a)}_{LN}(R)$ term, $\Sigma^{(b)}_{LN}(R)$ corresponds to $\sim |\mathbf{p}_1 + \mathbf{p}_2|^{-3}(p_1^{-3} + p_2^{-3})$ and finally $\Sigma^{(c)}_{LN}(R)$ is the loop contribution. In Appendix D we will show how these integrals may be partially done analytically.

• For a "linearised" LN density field, we obtain in a similar way as before the function $\Sigma_\delta(R) = \Sigma^{(a)}_\delta(R) + \Sigma^{(b)}_\delta(R)$, which is the sum of two contributions which originate from the two terms in the linear bispectrum $B^{(1)}_\delta$ in equation (48):

$$\Sigma_\delta(R) = 15\, \mathcal{K}^{(2)}_{xyxy}(\mathbf{p}_1, \mathbf{p}_2) \oplus p_1^{-2} p_2^{-2} |\mathbf{p}_1 + \mathbf{p}_2|^{-2}\, B^{(1)}_\delta(\mathbf{p}_1, \mathbf{p}_2)\ , \tag{56}$$

where

$$B^{(1)}_\delta(\mathbf{p}_1, \mathbf{p}_2) = A^2\, \Pi(\mathbf{p}_1, \mathbf{p}_2) \left\{ p_1\, p_2\, T(p_1)^2\, T(p_2)^2 + |\mathbf{p}_1 + \mathbf{p}_2|\, T(|\mathbf{p}_1 + \mathbf{p}_2|)^2 \left[p_1\, T(p_1)^2 + p_2\, T(p_2)^2 \right] \right\}\ . \tag{57}$$

Again, $\Sigma^{(a)}_\delta$ corresponds to the $p_1 p_2 T(p_1)^2 T(p_2)^2$ term and $\Sigma^{(b)}_\delta$ is the remaining contribution. Also these integrals may be reduced to simpler expressions analytically: the final formulae can be found in Appendix D.

Numerical values for the various Σs are shown in Fig. 1. Only positively skewed contributions are shown, negatively skewed have opposite sign. Some terms are logarithmically divergent for $p_m \to 0$, i.e., for *large* scales. This divergence can be traced back to the bispectrum. Note that the limits $p_m < p_{1,2} < p_M$ follow from restricting the initial scale-free power spectrum P_G to a finite interval; however, this generally sets no restrictions on $|\mathbf{p}_1 + \mathbf{p}_2|$ and the divergence occurs for elongated triangles $\mathbf{p}_1 \to -\mathbf{p}_2$ which can be controlled by the infra-red cut-off p_m in $\Pi(\mathbf{p}_1, \mathbf{p}_2)$. There are no ultra-violet divergences ($p_M \to \infty$) because such small scale interactions have been explicitly smoothed over. The different Σs have been scaled by appropriate powers of the mass variance $\sigma(R)$ to make them independent of the amplitude A of the power spectrum. In addition, the LN corrections have been scaled by their dependence on the dispersion σ_G of the underlying Gaussian model. Finally, some Σs have been scaled by powers of R to reduce the scale dependence of the plotted quantity. The $\Sigma^{(c)}_{LN}$ and Σ_{χ^2} contribution follow from 6D Monte-Carlo integrations and are less accurate than the others, which involve 3D numerical integrations.

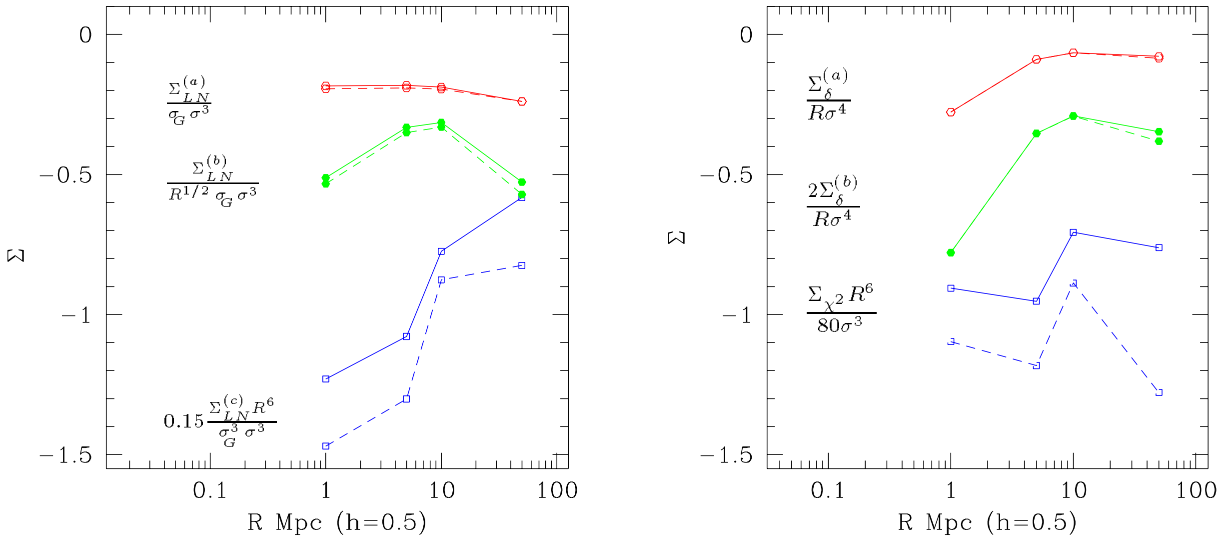

Figure 1. Correction terms to the linear angular momentum versus scale R, for several non-Gaussian statistics indicated in the panels (only the positively skewed ones are shown). The Σ–terms have been scaled by powers of the mass-variance $\sigma(R)$ to make them independent of the spectral normalisation and by powers of R to reduce the scale dependence of the plotted quantity. The lognormal terms have in addition been normalised by powers of σ_G. Full lines corresponds to the choice $p_m = 0.005\text{Mpc}^{-1}$ and dashed lines to $p_m = 0.01\text{Mpc}^{-1}$; $p_M = 2.5\text{Mpc}^{-1}$ for both, the scale R is in Mpc ($h = 0.5$).

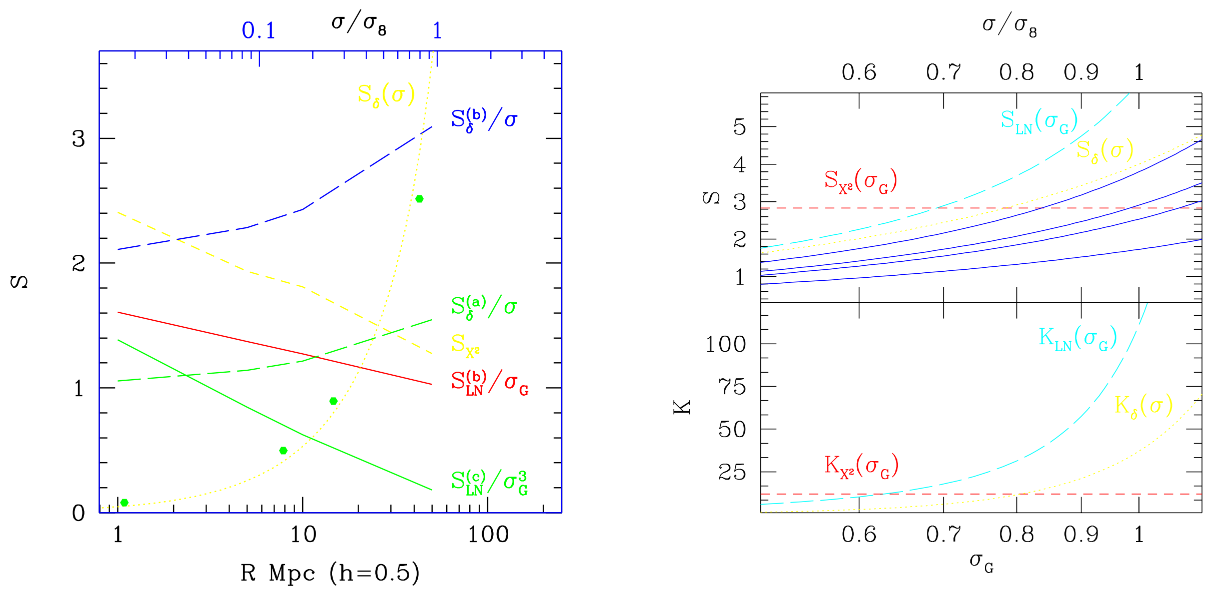

Figure 2. Left panel, lower scale: skewness S as a function of scale R (in Mpc, $h = 0.5$) for the different (positively skewed) statistics: S_{LN} and S_δ denote the density skewness for the lognormal gravitational potential and the lognormal density field, respectively. The individual contributions to S_{LN} are scaled by powers of σ_G, as indicated; S_δ is scaled by σ. The dotted line (upper scale) is S_δ versus σ/σ_8 for the LN density field taken from equation (43a) in Bernardeau & Kofman (1995). Filled diamonds show our determination of $S^{(a)}_\delta + S^{(b)}_\delta$ versus σ/σ_8 for comparison. Right panels, lower scale: skewness S (upper panel) and kurtosis K (lower panel) for the χ^2 (short dashed) and LN (long dashed) unfiltered models as function of σ_G. The χ^2 values are independent of σ_G. The *filtered* skewness for the LN gravitational potential distribution is plotted for comparison (upper panel, full lines) for filtering scales (from top to bottom) 1, 5, 10 and 50 Mpc, with $h = 0.5$. The dotted lines (upper scale) show the functions in equations (43a) and (43b) from Bernardeau & Kofman (1995) for skewness (upper panel) and kurtosis (lower panel) versus σ/σ_8 for the LN density field. For all curves, $p_m = 0.05\text{Mpc}^{-1}$, $p_M = 2.5\text{Mpc}^{-1}$, $h = 0.5$.

4 ANGULAR MOMENTUM AT THE MAXIMUM EXPANSION TIME

In this section we quantify the relative non-Gaussian perturbative corrections to the linear angular momentum for the different statistics by computing the parameter

$$\Upsilon(R) \equiv \frac{\langle \mathbf{L}^2 \rangle}{\langle \mathbf{L}^{(1)2} \rangle} - 1 = 2\, \frac{\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle}{\langle \mathbf{L}^{(1)2} \rangle} \,, \tag{58}$$

at the maximum expansion time, $D(\tau_R)\sigma(R) = 1$ (we computed the maximum expansion time for the spherical model; see Peebles 1969; see also Paper I).

• For the Chi–squared distribution we obtain

$$\Upsilon_{\chi^2}(R) = \frac{2\,\dot{E}}{\dot{D}D}\, \frac{\Sigma_{\chi^2}(R)}{\sigma(R)^3}\, [D(\tau_R)\sigma(R)] \approx -\frac{12}{7}\, \frac{\Sigma_{\chi^2}(R)}{\sigma(R)^3} \sim \pm\, 2.1\, (R/h^{-1}\mathrm{Mpc})^{-6} \,, \tag{59}$$

where the last value is estimated from Fig. (1). This correction is of $O(1)$ on galactic scales. We suggest that this indicates that higher-order terms in the perturbative expansion probably contribute significantly. We reiterate that the upper (lower) sign refers to positively (negatively) skewed distributions.

• For the LN distribution we obtain

$$\Upsilon_{LN}(R) = \frac{2\,\dot{E}}{\dot{D}D}\, \frac{\Sigma^{(a)}_{LN}(R) + \Sigma^{(b)}_{LN}(R) + \Sigma^{(c)}_{LN}(R)}{\sigma(R)^3}\, [D(\tau_R)\sigma(R)] \sim \pm[0.4\, \sigma_G + \sigma_G (R/h^{-1}\mathrm{Mpc})^{1/2} + 0.2\sigma_G^3 (R/h^{-1}\mathrm{Mpc})^{-6}], \tag{60}$$

which depends explicitly on σ_G, i.e., on the level of non-Gaussianity imposed on the initial state. Below we estimate an upper limit for σ_G and hence an upper limit to this correction.

• For the linearised LN density field, the scaling is rather different. This is because the degree of non-Gaussianity is determined directly by the amplitude of the power-spectrum and consequently the non-Gaussian correction term must also depend on that amplitude, parametrised for example by the mass-variance in spheres of $R_8 = 8h^{-1}$Mpc at the present time, σ_8. This complicates the dependence of the correction on the mass scale M. Approximating the scaling of σ with R for the standard CDM model as $\sigma(R)/\sigma_8 \approx (R_8/R)^{1.1+0.25\log(R/R_8)}$ and given the fact that the relevant Σs scale $\propto R\sigma(R)^4$, we obtain

$$\Upsilon_\delta(R) = \frac{2\,\dot{E}}{\dot{D}D}\, \frac{\Sigma^{(a)}_\delta(R) + \Sigma^{(b)}_\delta(R)}{\sigma(R)^3}\, [D(\tau_R)\sigma(R)] \approx \frac{2\,\dot{E}}{D\,\dot{D}}\, \frac{\Sigma^{(a)}_\delta(R) + \Sigma^{(b)}_\delta(R)}{R\sigma(R)^4}\, R\,\sigma_8 \left(\frac{R_8}{R}\right)^{1.1+0.25\log(R/R_8)} \sim \pm 6.3\,\sigma_8 \,, \tag{61}$$

where the numerical value is appropriate for scales $R = 0.5h^{-1}$ Mpc. Clearly, LN non-Gaussianity imposed directly on the density field can change quite drastically the amount of angular momentum induced on galactic scales. The absolute value of the correction depends on the normalisation of the power spectrum, through σ_8, which contrasts with the other distributions which are independent of σ_8.

4.1 Estimating the degree of non-Gaussianity

The relative spin correction Υ_{LN} for the case of a lognormally distributed gravitational potential φ depends on the free parameter σ_G, cfr equation (60), which fixes the degree of non-Gaussianity. An upper limit on the allowed degree of non-Gaussianity, and hence on σ_G, can be obtained by analysing the deviations from Gaussianity for the density field. This also allows us to show that distributions with similar skewness and kurtosis can nevertheless induce widely different corrections to the angular momentum.

In this spirit, we have computed numerically the skewness $S = \langle\delta^3\rangle/\langle\delta^2\rangle^{3/2}$ as a function of scale for all the considered non-Gaussian models starting from their respective bispectra: the result is shown in Fig. (2) which also shows the unfiltered kurtosis $K = \langle\delta^4\rangle/\langle\delta^2\rangle^2 - 3$. The dotted lines denote the expressions given by Bernardeau & Kofman (1995) for unfiltered skewness and kurtosis versus σ for the LN density field (their equations (43a) and (43b)). From Fig. (2) (left panel) it is clear that the skewness decreases weakly with scale for all models, except for the "linearised" LN density field which decreases more strongly: for $\sigma \lesssim 1$, $S_{LN} \approx 3\sigma$ (and $K_{LN} \approx 16\sigma^2$). Note that our estimate for the skewness for the *filtered* "linearised" LN density model versus σ agrees well with the value for the unfiltered one.

We can estimate an upper limit to σ_G for the gravitational potential LN model by requiring that the primordial skewness of the density field, $S_{\varphi_{LN}}$, is much smaller than the observed skewness, $S \approx 4$ on scales $R \approx 8\,h^{-1}$ Mpc. We thus set

$$S_{\varphi_{LN}}(\sigma_G) \approx 4\alpha \,, \tag{62}$$

with $\alpha \ll 1$. Using the approximations $S_{\varphi_{LN}}(\sigma_G) \approx 1.8\sigma_G + 0.6\sigma_G^3$ (from Fig. 2, for $\sigma_G \ll 1$) we get

$$\sigma_G \approx 2.22\alpha - 3.65\alpha^3 + 18\alpha^5 \quad \text{for} \;\; \alpha << 1 \,. \tag{63}$$

Taking $\alpha \sim 0.1$, we obtain $\sigma_G \sim 0.2$ from which $\Upsilon_{LN}(R = 0.5h^{-1}\mathrm{Mpc}) \sim \pm 0.33$. We conclude that in this approximation the non-Gaussian correction to the linear angular momentum for a LN potential is of the order of ≤ 33 per cent.

Summarising, it is clear that different underlying statistics give rise to differing corrections to the linear estimate of the angular momentum. For instance, the contribution of higher-order spin corrections appears to be non-negligible for χ^2 and for the "linearised" LN density field, whereas the perturbative expansion for the LN potential field seems to converge. Clearly, more detailed investigations of these non-Gaussian corrections seem warranted. Finally, note that in all cases positively skewed distributions increase the angular momentum of collapsing objects with respect to the linear estimate, and vice versa for negatively skewed ones.

5 SUMMARY AND CONCLUSIONS

In this paper we analysed the corrections to the linear growth of the tidal angular momentum $\mathbf{L}$ acquired by a proto-object (protogalaxy or protocluster) as a consequence of non-Gaussian initial conditions. We have used Lagrangian perturbation theory for the displacement field to obtain perturbative corrections to the linear angular momentum. Whereas the linear rms angular momentum involves integrals over the power spectrum alone – and hence is independent of the assumed statistics of the random field – the lowest-order perturbative corrections involve integrals over the bispectrum and so do depend on the underlying probability distribution. We showed that for a generic non-Gaussian distribution the corrections to the variance of the angular momentum grows $\propto t^{8/3}$, which contrasts with the Gaussian case for which the growth rate is $t^{10/3}$ (both for the Einstein-de Sitter universe). This is a consequence of the fact that the lowest order perturbative spin contribution in the non-Gaussian case arises from the third moment of the underlying density field, which is identically zero for a Gaussian field. In this formalism, the resulting expression for the variance of the spin factorises the same shape parameter as in the Gaussian case, since $\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle \propto \mu_1^2 - 3\mu_2$, where μ_1 and μ_2 are the first and second invariants of the inertia tensor of the collapsing object. This result does not depend on the assumed statistics, since it is a consequence of the isotropy of the Universe (see Appendix A).

The spin variance was evaluated explicitly as a function of scale for a variety of multiplicative non-Gaussian statistics, namely χ^2 and lognormal distributions imposed on the *gravitational potential* and "linearised" lognormal statistics assumed directly for the *density* field. We characterised these corrections in terms of $\Upsilon(R) \equiv \langle \mathbf{L}^2 \rangle / \langle {\mathbf{L}^{(1)}}^2 \rangle - 1$, i.e., in terms of the relative contribution of non-Gaussian corrections to the linear spin. In general, during the mildly non-linear regime, positively skewed distributions increase the angular momentum with respect to the linear term, and vice versa for negatively skewed ones. However, the convergence properties of the perturbative series depends strongly on the assumed statistics. We found that the χ^2 and "linearised" lognormal density statistics induce $O(1)$ corrections to the linear angular momentum, presumably indicating that higher-order terms, not discussed in this paper, are non-negligible. Consequently, angular momentum acquisition in models with this kind of underlying statistic appear analytically intractable. In contrast, the investigation of the lognormal distributions imposed on the *gravitational potential* seems more promising in this respect, since the lowest-order non-Gaussian spin correction are of order 30 per cent, suggesting convergence of the perturbative series. The calculations in this paper do not take into account non-gravitational processes or highly non-linear events which may play important roles in the formation of galaxies.

ACKNOWLEDGEMENTS

We acknowledge Dick Bond, George Efstathiou, Sabino Matarrese, Ravi Sheth, Radek Stompor and Rien van de Weygaert for discussions. Sabino Matarrese is particularly thanked for allowing to reproduce in Appendix B some of his unpublished notes. PC is personally grateful to Michal Chodorowski, Ewa Łokas and Lauro Moscardini for permitting us to use some preliminary results of a non–Gaussian collaboration still in progress, and for their patience for the continuous delays caused to the common projects. Constructive remarks of the Referee improved this work. PC and TT were supported by the EEC Human Capital and Mobility Programme under contracts CT930328 and CT941463 respectively.

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APPENDIX A

In this first appendix we report the details of the derivation of equation (21) for the spin correction $\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle$, namely

$$\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle = \frac{2}{15}\,\dot{D}(\tau)\dot{E}(\tau)\,(\mu_1^2 - 3\mu_2)\,\Sigma(R)\,, \tag{64}$$

where the wave-vector contribution $\Sigma(R)$ is defined by

$$\Sigma(R) \equiv 15\,\langle \mathcal{D}^{(1)}_{xy}\,\mathcal{D}^{(2)}_{xy} \rangle = 15\,\mathcal{K}^{(2)}_{xyxy} \oplus B^{(1)}_{\psi}\,. \tag{65}$$

We will show that this expression is a direct consequence of the isotropy of the universe.

Our starting point is the general expression [see equation (18)]

$$\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle = \dot{D}(\tau)\,\dot{E}(\tau)\,\epsilon_{\alpha\beta\gamma}\,\epsilon_{\alpha\beta'\gamma'}\,\mathcal{J}_{\sigma\gamma}\,\mathcal{J}_{\sigma'\gamma'}\,\langle \mathcal{D}^{(1)}_{\beta\sigma}\,\mathcal{D}^{(2)}_{\beta'\sigma'} \rangle\,. \tag{66}$$

The Levi-Civita tensors may be written as $\epsilon_{\alpha\beta\gamma}\,\epsilon_{\alpha\beta'\gamma'} = \delta_{\beta\beta'}\delta_{\gamma\gamma'} - \delta_{\beta\gamma'}\delta_{\beta'\gamma}$, which leads to:

$$\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle = \dot{D}(\tau)\,\dot{E}(\tau)\,\mathcal{J}_{\sigma\gamma}\left[\mathcal{J}_{\sigma'\gamma}\,\langle \mathcal{D}^{(1)}_{\beta\sigma}\,\mathcal{D}^{(2)}_{\beta\sigma'} \rangle - \mathcal{J}_{\sigma'\beta}\,\langle \mathcal{D}^{(1)}_{\beta\sigma}\,\mathcal{D}^{(2)}_{\gamma\sigma'} \rangle\right]. \tag{67}$$

Without loss of generality, we now assume that the chosen frame of reference coincides with the eigenframe of the inertia tensor, in which case $\mathcal{J}_{\alpha\beta} = \iota_\alpha \delta_{\alpha\beta}$, where ι_α are the eigenvalues of $\mathcal{J}$ (no summation over α in this specific case!). In this particular reference frame, we get

$$\begin{aligned}
\langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle &= \dot{D}(\tau)\,\dot{E}(\tau)\,(\iota_\sigma^2 - \iota_\sigma\iota_\beta)\,\langle \mathcal{D}^{(1)}_{\beta\sigma}\,\mathcal{D}^{(2)}_{\beta\sigma} \rangle \qquad (\beta \neq \sigma)\\
&= \dot{D}(\tau)\,\dot{E}(\tau)\left[2(\iota_x^2 + \iota_y^2 + \iota_z^2)\,\langle \mathcal{D}^{(1)}_{xy}\,\mathcal{D}^{(2)}_{xy} \rangle - 2(\iota_x\iota_y + \iota_x\iota_z + \iota_y\iota_z)\,\langle \mathcal{D}^{(1)}_{xy}\,\mathcal{D}^{(2)}_{xy} \rangle\right]\\
&= 2\dot{D}(\tau)\,\dot{E}(\tau)\,(\mu_1^2 - 3\mu_2)\,\langle \mathcal{D}^{(1)}_{xy}\,\mathcal{D}^{(2)}_{xy} \rangle\,,
\end{aligned} \tag{68}$$

where $\mu_1 \equiv \iota_x + \iota_y + \iota_z$ and $\mu_2 \equiv \iota_x\iota_y + \iota_x\iota_z + \iota_y\iota_z$ are the first and the second invariant of the inertia tensor $\mathcal{J}$. The latter equation (68) was used in Section 3.2

APPENDIX B

In this Appendix we describe how to obtain the expression of the Chi–squared power spectrum in equation (32), using the same notation as in Section 3.1. The power spectrum of the 2–point correlation function of the field $\varphi = \varphi_\circ\,G^2$ is given by

$$P_\varphi(p) = 2\varphi_\circ^2 \int d\mathbf{q}\,[\xi_G(q)]^2\,\mathrm{e}^{-i\mathbf{q}\cdot\mathbf{p}} = \frac{\varphi_\circ^2}{4\pi^3}\int d\mathbf{k}\,P_G(k)\,P_G(|\mathbf{k}-\mathbf{p}|)\,. \tag{69}$$

Changing variable from azimuthal angle θ (using spherical coordinates in $\mathbf{k}$) to $\ell = |\mathbf{k}-\mathbf{p}|$ simplifies the previous integral to:

$$P_\varphi(p) = \frac{\varphi_\circ^2}{2\pi^2\,p}\int_0^\infty dk\,k\,P_G(k)\int_{|p-k|}^{p+k} d\ell\,\ell\,P_G(\ell)\,. \tag{70}$$

Let us now consider the flicker–noise power spectrum for G defined in equation (31) and introduce the parameters $\epsilon \equiv k_m/p$ and $Q \equiv k_M/p$. The condition $k_m \le p \le k_M$ then translates into $\epsilon \le 1$ and $Q \ge 1$ and the previous integral simplifies further to

$$P_\varphi(p) = \frac{\varphi_\circ^2}{2\pi^2\,p^3}\int_\epsilon^Q dk\,k^{-2}\int_{\mathrm{Max}\{\epsilon,|k-1|\}}^{\mathrm{Min}\{Q,k+1\}} d\ell\,\ell^{-2}\,. \tag{71}$$

In the limit $Q \to \infty$ one has

$$P_\varphi(p) = \frac{\varphi_\circ^2}{2\pi^2\,p^3}\left[\int_\epsilon^1 \frac{dk}{k^2}\,\frac{1}{\mathrm{Max}\{\epsilon,1-k\}} + \int_1^\infty \frac{dk}{k^2}\,\frac{1}{\mathrm{Max}\{\epsilon,k-1\}} - \int_\epsilon^\infty \frac{dk}{k^2}\,\frac{1}{k+1}\right], \tag{72}$$

and two expressions for P_φ may be recovered: for $p \ge 2\,k_m$, i.e. $\epsilon \le 1/2$,

$$P_\varphi(p) = \frac{\varphi_\circ^2}{2\pi^2\,p^3}\,2\ln\left(\frac{1-\epsilon^2}{\epsilon^2}\right)\,, \tag{73}$$

while for $k_m \le p \le 2k_m$, i.e. $\epsilon \ge 1/2$, the power spectrum reduces to

$$P_\varphi(p) = \frac{\varphi_\circ^2}{2\pi^2\,p^3}\left[2\ln\left(\frac{1+\epsilon}{\epsilon}\right) + \left(\frac{1-\epsilon}{\epsilon}\right)^2 - 1\right]. \tag{74}$$

Substituting $\epsilon = k_m/p$ into the latter equations leads to the compact expression for the Chi–squared power spectrum in equation (32).

APPENDIX C

In this third Appendix we summarise those properties of the Chi–squared and lognormal distributions which are relevant to our analysis. To indicate the non–Gaussian field, we employ the same symbol φ used in the main text for the primordial gravitational potential; however, the results derived below hold for any non-Gaussian Chi–squared or lognormal field. The reader interested in a more comprehensive treatment is addressed to Kendall & Stuart (1977).

C.1 Chi–squared distribution

Let G be a Gaussian variable with zero mean and variance σ_G^2. The variable $\varphi \equiv \varphi_\circ G^2$ has a Chi-squared distribution, with characteristic function

$$\Phi_\varphi(t) = \frac{1}{\sqrt{1 - 2i\,\varphi_\circ\,\sigma_G^2\,t}} \; . \tag{75}$$

The moments μ_n of φ about the origin $\varphi = 0$ can be obtained by evaluating successive derivatives of the characteristic function $\Phi_\varphi(t)$ at $t = 0$:

$$\mu_n = \langle \varphi^n \rangle = (-i)^n \left. \frac{d^n \Phi_\varphi(t)}{dt^n} \right|_{t=0} \; . \tag{76}$$

Explicitly, the first few moments are found to be:

$$\mu_1 = \varphi_\circ\,\sigma_G^2 \; ; \qquad \mu_2 = 3\,\varphi_\circ^2\,\sigma_G^4 \; ; \qquad \mu_3 = 15\,\varphi_\circ^3\,\sigma_G^6 \; . \tag{77}$$

The reduced moments about the mean, or cumulants, are ($\kappa_1 = 0$)

$$\kappa_2 = \langle (\varphi - \langle\varphi\rangle)^2 \rangle = \mu_2 - \mu_1^2 = 2\,\varphi_\circ^2\,\sigma_G^4 \; , \tag{78}$$

$$\kappa_3 = \langle (\varphi - \langle\varphi\rangle)^3 \rangle = \mu_3 - 3\,\mu_1\,\mu_2 + 2\,\mu_1^3 = 8\,\varphi_\circ^3\,\sigma_G^6 \; , \tag{79}$$

The cumulants κ_n with $n > 2$ vanish, by definition, for a Gaussian distribution. The deviation from non-Gaussianity of a field φ with given dispersion σ_φ^2 obtained in this way cannot be tuned by changing σ_G of the underlying Gaussian field. Indeed, since $\sigma_\varphi^2 = 3\varphi_0^2\sigma_G^4$ is assumed to be constant, we find $\kappa_2 = 2\sigma_\varphi^2/3$ and $\kappa_3 = 8\sigma_\varphi^3/3$, independent of σ_G. Similar relations hold true for all higher moments as well. The same remark applies to all fields $\varphi \propto G^n$, obtained from powers of an underlying Gaussian field. In contrast, we will show below that for LN statistics, the deviation from Gaussianity can be tuned by varying σ_G.

The characteristic function of φ is useful to calculate the moments of its one-point PDF. To compute the *correlation functions* of the field φ, it is convenient to work with the characteristic function $\Phi_G^{(n)}$ of the multivariate Gaussian probability distribution of the field $G(\mathbf{q})$, where the dependence of G on the 'spatial' variable $\mathbf{q}$ is now emphasized. It is given by:

$$\Phi_G^{(n)}(\mathbf{t}) = \exp\left[-\frac{1}{2}\,\mathbf{t}^T \mathcal{C}\,\mathbf{t} \right] \; , \tag{80}$$

where $\mathbf{t} \equiv (t_1, \ldots, t_n)$, $\mathbf{t}^T$ denotes the transpose of $\mathbf{t}$ and $\mathcal{C}$ is the covariance matrix of the Gaussian field G, whose elements are

$$\langle G(\mathbf{q}_i)\,G(\mathbf{q}_j) \rangle = \xi_G(q_{ij}) \tag{81}$$

and $q_{ij} \equiv |\mathbf{q}_i - \mathbf{q}_j|$. Here, $\xi_G(q)$ is the covariance (correlation) function of the Gaussian field G with $\xi_G(0) = \sigma_G^2$. By definition, the 2-point correlation function of the field φ is

$$\xi_\varphi(q) \equiv \langle (\varphi_1 - \langle\varphi\rangle)\,(\varphi_2 - \langle\varphi\rangle) \rangle = \langle \varphi_1\,\varphi_2 \rangle - \langle\varphi\rangle^2 \; , \tag{82}$$

where $\varphi_i \equiv \varphi(\mathbf{q}_i)$. We have

$$\varphi_\circ^{-2} \langle \varphi_1\,\varphi_2 \rangle = \langle G_1^2\,G_2^2 \rangle = \left. \frac{\partial^4 \Phi_G^{(2)}}{\partial t_1^2\,\partial t_2^2} \right|_{\mathbf{t}=\mathbf{0}} = \sigma_G^4 + 2\,\xi_G(q)^2 \; . \tag{83}$$

Combining equations (82) and (83), we obtain the following relation between the correlation function of φ and that of G:

$$\xi_\varphi(q) = 2\,\varphi_\circ^2\,\xi_G(q)^2 \; . \tag{84}$$

In a similar way, using the characteristic function of order n of the multivariate Gaussian distribution function, one may derive the higher-order correlation functions of the field φ. Thus, the 3-point connected correlation function is

$$\zeta_\varphi(123) \equiv \langle (\varphi_1 - \langle\varphi\rangle)\,(\varphi_2 - \langle\varphi\rangle)\,(\varphi_3 - \langle\varphi\rangle) \rangle = 8\,\varphi_\circ^3\,\xi_G(12)\,\xi_G(13)\,\xi_G(23) \; , \tag{85}$$

where $\xi_G(ij) \equiv \xi_G(\mathbf{q}_i, \mathbf{q}_j)$, and so on. Note that, from equations (84) and (85), one has $\kappa_2 = \xi_\varphi(0) = 2\varphi_\circ^2\,\sigma_G^4$ and $\kappa_3 = \zeta_\varphi(0) = 8\varphi_\circ^3\,\sigma_G^6$, in agreement with equations (78) and (79). The bispectrum B_φ may be recovered by performing

the Fourier transform of equation (85), which gives the expression reported in equation (33). We stress once more the absence of any explicit dependence on the underlying variance σ_G^2.

C.2 Lognormal distribution

Let us again suppose that G is a normal variable with zero mean and variance σ_G^2. The variable $\varphi \equiv \varphi_\circ \, \mathrm{e}^G$ is said to be lognormal distributed. The 2-point correlation function of the field φ, defined in equation (82), may be obtained from the correlation ξ_G of the field G via the bivariate characteristic $\Phi_G^{(2)}$. From equations (80) and (81) we get

$$\varphi_\circ^{-2} \, \langle \varphi_1 \, \varphi_2 \rangle = \langle \mathrm{e}^{G_1+G_2} \rangle = \Phi_G^{(2)}(-i,-i) = \mathrm{e}^{\sigma_G^2} \, \mathrm{e}^{\xi_G(q)} \; . \tag{86}$$

Moreover, $\varphi_\circ^{-1} \, \langle \varphi \rangle = \Phi_G^{(1)}(-i) = \mathrm{e}^{\sigma_G^2/2}$. Combining these expressions one gets again a relation between the two 2-point correlation functions:

$$\xi_\varphi(q) = \langle \varphi \rangle^2 \, \left[\mathrm{e}^{\xi_G(q)} - 1 \right] \; . \tag{87}$$

In a similar fashion, one can obtain the 3–point correlation function, defined in equation (85). Since

$$\zeta_\varphi(123) = \langle \varphi_1 \, \varphi_2 \, \varphi_3 \rangle - \langle \varphi \rangle \, \langle \varphi_1 \, \varphi_2 \rangle - \langle \varphi \rangle \, \langle \varphi_2 \, \varphi_3 \rangle - \langle \varphi \rangle \, \langle \varphi_1 \, \varphi_3 \rangle + 2 \, \langle \varphi \rangle^3 \; , \tag{88}$$

one obtains

$$\langle \varphi_1 \varphi_2 \varphi_3 \rangle = \langle \varphi \rangle^3 + \langle \varphi \rangle \, [\xi_\varphi(12) + \xi_\varphi(13) + \xi_\varphi(23)] + \zeta_\varphi(123) \; . \tag{89}$$

On the other hand, the left hand side of the latter equation may be written as

$$\begin{aligned} \langle \varphi_1 \, \varphi_2 \, \varphi_3 \rangle &= \varphi_\circ^3 \, \langle \mathrm{e}^{G_1+G_2+G_3} \rangle = \varphi_\circ^3 \, \Phi_G^{(3)}(-i,-i,-i) \\ &= \langle \varphi \rangle^3 \, \mathrm{e}^{\xi_G(12)+\xi_G(13)+\xi_G(23)} = \langle \varphi \rangle^3 \, [1 + \langle \varphi \rangle^{-2} \xi_\varphi(12)] \, [1 + \langle \varphi \rangle^{-2} \xi_\varphi(13)] \, [1 + \langle \varphi \rangle^{-2} \xi_\varphi(23)] \\ &= \langle \varphi \rangle^{-3} \, \xi_\varphi(12) \, \xi_\varphi(13) \, \xi_\varphi(23) + \langle \varphi \rangle^{-1} \, [\xi_\varphi(12) \, \xi_\varphi(13) + \xi_\varphi(12) \, \xi_\varphi(23) + \xi_\varphi(13) \, \xi_\varphi(23)] \\ &+ \langle \varphi \rangle \, [\xi_\varphi(12) + \xi_\varphi(13) + \xi_\varphi(23)] + \langle \varphi \rangle^3 \; , \end{aligned} \tag{90}$$

and, from (89), one recovers a Kirkwood-like expansion for the 3-point connected correlation function:

$$\zeta_\varphi(123) = \langle \varphi \rangle^{-1} \, [\xi_\varphi(13) \, \xi_\varphi(23) + \xi_\varphi(12) \, \xi_\varphi(13) + \xi_\varphi(12) \, \xi_\varphi(23)] + \langle \varphi \rangle^{-3} \, [\xi_\varphi(12) \, \xi_\varphi(13) \, \xi_\varphi(23)] \; . \tag{91}$$

The first few cumulants κ_n of this lognormal distribution are:

$$\kappa_2 = \xi_\varphi(0) = \varphi_\circ^2 \, \mathrm{e}^{\sigma_G^2} \, \left(\mathrm{e}^{\sigma_G^2} - 1 \right) \; , \tag{92}$$

$$\kappa_3 = \zeta_\varphi(0) = \varphi_\circ^3 \, \mathrm{e}^{3\sigma_G^2/2} \, \left(\mathrm{e}^{\sigma_G^2} - 1 \right)^2 \, \left(\mathrm{e}^{\sigma_G^2} + 2 \right) \; . \tag{93}$$

Taking again $\sigma_\varphi^2 = \kappa_2$ constant by setting $\varphi_0^2 = \kappa_2 / e^{\sigma_G^2} \, (e^{\sigma_G^2} - 1)$ where σ_G is now a free parameter, one can write $\kappa_3 = \kappa_2^{3/2} \, (e^{\sigma_G^2} + 1)(e^{\sigma_G^2} - 1)^{1/2}$, from which it is clear that the skewness of φ can be tuned independently of its variance. Clearly, this is also true for higher moments. In other words, the deviation from Gaussianity of the LN φ can be tuned by changing σ_G. In particular, in the limit $\sigma_G^2 \to 0$ the lognormal distribution tends to a Gaussian. The bispectrum B_φ of the lognormal variable φ may be derived by Fourier transforming the expression (91) (Catelan, Chodorowski, Łokas and Moscardini 1997). The final result is reported in equation (39). An explicit dependence on the underlying variance σ_G^2 can be noted in equation (41), which holds in the regime $\sigma_G^2 \lesssim 1$.

APPENDIX D

In this last appendix we want to show how to perform analytically some of the integrations when computing the spin correction $\Sigma(R)$ for the different non-Gaussian fields we have considered. The compact expression for $\Sigma(R)$ in terms of the kernel $\mathcal{K}_{xyxy}^{(2)}$ enables us to reduce easily the number of numerical integrations by exploiting rotational invariance of the integrals.

• We start by considering the simplest case, the $\Sigma_{LN}(R)$ correction for the LN gravitational potential φ:

$$\Sigma_{LN}(R) = 15 \, \mathcal{K}_{xyxy}^{(2)} \oplus B_\psi^{(1)} = \Sigma_{LN}^{(a)}(R) + \Sigma_{LN}^{(b)}(R) + \Sigma_{LN}^{(c)}(R) \; . \tag{94}$$

As discussed in the main text, the three contributions $\Sigma_{LN}^{(h)}(R)$ originate from the Kirkwood relation of the LN bispectrum $B_\psi^{(1)}$ [see equations (29) and (39)]. Let us consider the first of these three contributions:

$$\Sigma_{LN}^{(a)}(R) = 15 \, \langle \varphi \rangle^{-1} \, \mathcal{K}_{xyxy}^{(2)}(\mathbf{p}_1, \mathbf{p}_2) \oplus \mathcal{T}(p_1) \, \mathcal{T}(p_2) \, \mathcal{T}(|\mathbf{p}_1 + \mathbf{p}_2|) \, P_\varphi(\mathbf{p}_1) \, P_\varphi(\mathbf{p}_2) \; . \tag{95}$$

which can be written explicitly as:

$$\Sigma^{(a)}_{LN}(R) = -\frac{15}{\langle\varphi\rangle}\int\frac{d\mathbf{p}_1\,d\mathbf{p}_2}{(2\pi)^6}\,\frac{(\mathbf{p}_1+\mathbf{p}_2)^2_x\,(\mathbf{p}_1+\mathbf{p}_2)^2_y}{|\mathbf{p}_1+\mathbf{p}_2|^2}\,\kappa^{(2)}(\mathbf{p}_1,\mathbf{p}_2)\,[\widetilde{W}(|\mathbf{p}_1+\mathbf{p}_2|R)]^2\,\mathcal{T}(p_1)\,\mathcal{T}(p_2)\,\mathcal{T}(|\mathbf{p}_1+\mathbf{p}_2|)\,P_\varphi(\mathbf{p}_1)\,P_\varphi(\mathbf{p}_2)\,. \quad (96)$$

We now use the relation $P_\varphi(p) \approx A_\varphi F_\varphi(p) p^{-3}\Pi(p)$, with $F_\varphi = 1$ for the LN case (and hence $\mathcal{T}(p) = T(p)$) and in addition $A_\varphi = 2\pi^2\sigma^2_G\,\langle\varphi\rangle^2/\ln(p_M/p_m) \approx A$. Furthermore, without loss of generality, we assume that the vector $\mathbf{p}_1$ is chosen along the z-direction, in which case the previous expression simplifies to:

$$\Sigma^{(a)}_{LN}(R) = -15\,\frac{A^2_\varphi}{\langle\varphi\rangle}\int\frac{d\mathbf{p}_1 d\mathbf{p}_2}{(2\pi)^6}\,\frac{p^2_{2x}\,p^2_{2y}}{|\mathbf{p}_1+\mathbf{p}_2|^2}\,\kappa^{(2)}(\mathbf{p}_1,\mathbf{p}_2)\,\left[\widetilde{W}(|\mathbf{p}_1+\mathbf{p}_2|R)\right]^2 T(p_1)\,T(p_2)\,T(|\mathbf{p}_1+\mathbf{p}_2|)\,p_1^{-3}\,p_2^{-3}\Pi(\mathbf{p}_1,\mathbf{p}_2)\,. \quad (97)$$

Note that the integrand does not depend on the azimuthal angle θ_1 nor on the precessional angle ϕ_1; in addition, the ϕ_2-integration may be done analytically. The final result may be cast in the form:

$$\begin{aligned}\Sigma^{(a)}_{LN}(R) = &-\frac{15}{8(2\pi)^4}\left[\pm A^{3/2}\sqrt{\frac{2\pi^2\sigma^2_G}{\ln(p_M/p_m)}}\right]\int_0^\infty dp_1\,T(p_1)\int_0^\infty dp_2\,p_2^4\,T(p_2)\int_0^\pi d\theta_2\,(\sin\theta_2)^7\\ &\times\left(p_1\,p_2^{-1}+p_2\,p_1^{-1}+2\cos\theta_2\right)^{-1}T(|\mathbf{p}_1+\mathbf{p}_2|)\left[\widetilde{W}(|\mathbf{p}_1+\mathbf{p}_2|R)\right]^2\Pi(\mathbf{p}_1,\mathbf{p}_2)\,, \end{aligned} \quad (98)$$

where θ_2 is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$. The upper (lower) sign is for positively (negatively) skewed LN distributions. The remaning integral needs to be computed numerically.

The other two contributions may be calculated in a similar way. The second contribution is

$$\begin{aligned}\Sigma^{(b)}_{LN}(R) = &-\frac{15}{4(2\pi)^4}\left[\pm A^{3/2}\sqrt{\frac{2\pi^2\,\sigma^2_G}{\ln(p_M/p_m)}}\right]\int_0^\infty dp_1\,p_1^{-3/2}\,T(p_1)\int_0^\infty dp_2\,p_2^{11/2}\,T(p_2)\int_0^\pi d\theta_2\,(\sin\theta_2)^7\\ &\times\left(p_1\,p_2^{-1}+p_2\,p_1^{-1}+2\cos\theta_2\right)^{-5/2}T(|\mathbf{p}_1+\mathbf{p}_2|)\left[\widetilde{W}(|\mathbf{p}_1+\mathbf{p}_2|R)\right]^2\Pi(\mathbf{p}_1,\mathbf{p}_2)\,. \end{aligned} \quad (99)$$

The computation of the third contribution is the more involved, since it corresponds to a 9 dimensional integration. Recalling that the bispectrum B_φ depends only on the moduli of the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ and the mutual angle $\theta_{12} = \theta_2$, one has

$$\begin{aligned}\Sigma^{(c)}_{LN}(R) = &-\frac{15}{8(2\pi)^7}\left[\pm A^{3/2}\left(\frac{2\pi^2\,\sigma^2_G}{\ln(p_M/p_m)}\right)^{3/2}\right]\int_0^\infty dp_1\,p_1^{3/2}\,T(p_1)\int_0^\infty dp_2\,p_2^{11/2}\,T(p_2)\int_0^\pi d\theta_2\,(\sin\theta_2)^7\\ &\times\left(p_1\,p_2^{-1}+p_2\,p_1^{-1}+2\cos\theta_2\right)^{-1}T(|\mathbf{p}_1+\mathbf{p}_2|)\left[\widetilde{W}(|\mathbf{p}_1+\mathbf{p}_2|R)\right]^2\mathcal{B}(p_1,p_2;\theta_2)\,, \end{aligned} \quad (100)$$

where

$$\begin{aligned}\mathcal{B}(p_1,p_2;\theta_2) \equiv &\int_0^\infty dp_3\,p_3^{-4}\int_0^\pi d\theta_3\,\sin\theta_3\left(p_1\,p_3^{-1}+p_3\,p_1^{-1}+2\cos\theta_3\right)^{-3/2}\\ &\times\int_0^{2\pi} d\phi_3\left[p_2\,p_3^{-1}+p_3\,p_2^{-1}-2(\cos\theta_3\cos\theta_2+\sin\theta_3\sin\theta_2\cos\phi_3)\right]^{-3/2}\Pi(\mathbf{p}_1,\mathbf{p})\Pi(\mathbf{p}_2,-\mathbf{p})\Pi(\mathbf{p}_1,\mathbf{p}_2)\,. \end{aligned} \quad (101)$$

This leaves a 6D numerical integration to be performed.

• The Chi-squared correction $\Sigma_{\chi^2}(R)$ may be calculated in the same fashion as the contribution $\Sigma^{(c)}_{LN}(R)$; the result is

$$\begin{aligned}\Sigma_{\chi^2}(R) = &-\frac{15}{(2\pi)^4}\left[\pm\left(\frac{A}{2}\right)^{3/2}\right]\int_0^\infty dp_1\,p_1^{3/2}\,\mathcal{T}(p_1)\int_0^\infty dp_2\,p_2^{11/2}\,\mathcal{T}(p_2)\int_0^\pi d\theta_2\,(\sin\theta_2)^7\\ &\times\left(p_1\,p_2^{-1}+p_2\,p_1^{-1}+2\cos\theta_2\right)^{-1}\mathcal{T}(|\mathbf{p}_1+\mathbf{p}_2|)\left[\widetilde{W}(|\mathbf{p}_1+\mathbf{p}_2|R)\right]^2\mathcal{B}(p_1,p_2;\theta_2)\,\Pi(\mathbf{p}_1,\mathbf{p})\Pi(\mathbf{p}_2,-\mathbf{p})\,, \end{aligned} \quad (102)$$

where $\mathcal{T}(p) \equiv T(p)[\beta(p) + 2\ln(1+p/p_m) - 1]^{-1/2}$.

• For the "linearised" LN density field one can perform calculations in a similar way, which lead to equation (56), where the function $\Sigma_\delta(R) = \Sigma^{(a)}_\delta(R) + \Sigma^{(b)}_\delta(R)$ is the sum of two contributions originated by the two terms of the linear bispectrum $B^{(1)}_\delta$ in equation (48):

$$\begin{aligned}\Sigma^{(a)}_\delta(R) = &-\frac{15}{8}\,\frac{A^2}{(2\pi)^4}\int_0^\infty dp_1\,p_1\,T(p_1)^2\int_0^\infty dp_2\,p_2^5\,T(p_2)^2\int_0^\pi d\theta_2\,(\sin\theta_2)^7\\ &\times\left(p_1\,p_2^{-1}+p_2\,p_1^{-1}+2\cos\theta_2\right)^{-2}\left[\widetilde{W}(|\mathbf{p}_1+\mathbf{p}_2|R)\right]^2\Pi(\mathbf{p}_1,\mathbf{p}_2)\,. \end{aligned} \quad (103)$$

The second contribution is

$$\Sigma^{(b)}_\delta(R) = -\frac{15}{8}\,\frac{A^2}{(2\pi)^4}\int_0^\infty dp_1\,p_1^{3/2}\,T(p_1)^2\int_0^\infty dp_2\,p_2^{9/2}\int_0^\pi d\theta_2\,(\sin\theta_2)^7$$

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$$\times \left(p_1\, p_2^{-1} + p_2\, p_1^{-1} + 2\cos\theta_2\right)^{-3/2} \left[T(|\mathbf{p}_1 + \mathbf{p}_2|)\right]^2 \left[\widetilde{W}(|\mathbf{p}_1 + \mathbf{p}_2|R)\right]^2 \Pi(\mathbf{p}_1, \mathbf{p}_2)\ . \tag{104}$$